Logic & Critical Reasoning

Conceptual Foundations and Techniques of Evaluation

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Acknowledgements and Sources

I decided to write this work in the summer of 2006, after my first year of teaching at SJSU. I don't know what prompted it other than the thought that I could do a decent job, and that I needed something to do while I was watching the World Cup, hence, the many translation exercises having to do with soccer.

Although a set of notes came out of that summer's work, a thorough draft was not completed. Over the past 4 years I have refined those notes many times in my Philosophy 57 Logic and Critical Thinking course. And Andrew Erickson has provided the most comprehensive feedback, editing, and revisions. This version would not be possible without him, and is far better than any version I had drafted.

The other two people without whom this book would not have been completed are: **Mike Cole** and **Richard Glatz**. Both are far better logicians than I will ever be. And none of the mistakes here are due to them. I have over many years, even prior to the conception of this work, talked with them about different ways of presenting parts of logic. Many of the sections in this work come either directly from conversations with them or altered versions of work they have sent me. Without a doubt Mike and Richard are my biggest influences in teaching logic and critical thinking.

Finally, in coming up with examples for this book I have looked at Patrick Hurley's A Concise Introduction to Logic, Irving M. Copi & Carl Cohen's Logic, and C. Stephen Layman's The Power of Logic. In each case I have looked over many different editions. Perhaps the areas that are most sensitive to those works are the exercises on informal fallacies and some of the exercises on natural deduction.

While I think each of the works above to be an excellent introduction to logic, this book was largely written so as to serve as a cheaper alternative to the other works. I have dispensed with much of the material covered in those books, which many teachers do not include in their course.

Table of Contents:

Guides and Exercises:

Unit 1
Arguments and Techniques of Informal Analysis

Chapter 1
Identifying Arguments

1.1 The elements of an argument

Logic and critical reasoning comprise that domain of inquiry where arguments are the basic objects of investigation. From a logical point of view arguments are taken to be a specific kind of entity, and not necessarily something that people disagree or agree with. Often when the term 'argument' is used in public discourse one intends partially to signify that some people disagree about something. However, from within critical reasoning and logic arguments are to be distinguished from disagreements. Two people can disagree about something without arguing. For example, Mary and Sandra may disagree over whether humans have a right to healthcare without either of them ever offering any reasons in favor of the position they hold. Likewise, one can construct an argument without anyone disagreeing with the argument precisely because no one has ever considered the topic or because no one disagrees with what is said in the argument. Moreover, disagreement has no essential connection to what an argument is. It is neither necessary nor sufficient that two or more people disagree about something for an argument to be present.

As defined by logic an *argument* is simply a sequence of statements in which one statement in the sequence is supported by the other statements in the sequence. The statement that is supported is called the *conclusion*, and the statements that offer support or reasons in favor of believing the conclusion are called the *premises*. What is encapsulated in an argument is that reasons are offered in favor of a specific statement. The reasons are contained in the premises.

An argument makes two distinct claims:

- A *factual* claim – that the premises of the argument are true.
- An *inferential* claim – that the premises support the conclusion.

Ex. 1.1.1

All humans are mortal. Socrates is a human. So, Socrates is mortal.

Ex. 1.1.2

If Hezbollah bombs northern Israel, then Israel will attack southern Lebanon. Hezbollah will bomb northern Israel. So, Israel will attack southern Lebanon.

In examples 1.1.1 and 1.1.2 the first two sentences in each sequence of sentences is a premise. The premises are offered in favor of the conclusion, which is the final sentence in each sequence.

In understanding arguments it is important to recognize that only certain sequences of sentences can be arguments. An argument defined as a set of premises offered in support of a conclusion requires that all of the premises and the conclusion be *statements*. In a natural language, such as English, the utterances we make and the sentences we write generally fall into four categories: questions, commands, exclamations, and declarations. Statements form a sub-class of declarative sentences.

(a) Questions:
  "What time is it?" "Where is the mall?" "How tall are you?"

(b) Commands:
  "Close the door." "Find the mall." "Get the milk."

(c) Exclamations:
  "Ouch!" "#&@@#$%!" "MOFO!"

(d) Statements:
  "John is tall" "All dogs are animals" "9 is greater than 7"

The distinguishing feature of statements, in comparison to questions, commands, and exclamations is that statements express *propositions*. Propositions are the kinds of things that can be true or false, and so statements are derivatively the kinds of things that can be true or false. Propositions have *truth-evaluable content*. Their content is of the type where it is appropriate to speak of truth and falsity. For example, it is either true or false that John is tall; that all dogs are animals; and that 9 is greater than 7. By contrast, exclamations, commands and questions cannot be true or false. For example, it is true or false that the door is closed or open, but in response to the command "close the door" one either obeys it or disobeys it. It makes no sense to say in response to the command "Close the door!" "True." Likewise, in response to the question 'what time is it?' one answers with the time, but not with 'true' or 'false'. Questions, commands, and exclamations do not have truth-evaluable content.

For the purpose of clarifying this even more, we can introduce the concept of a *category mistake*. A category mistake occurs when a property appropriate for one category is applied to a category to which it does not apply.

Ex. 1.1.3

"2 has parents from America."

Taking '2' to be the name of the number 2, and not of a human being or an animal, it is incoherent to say that the number two has parents from America. The reason why is that numbers –in the literal sense- don't have biological parents or come from specific nations. The property of having a parent applies properly and appropriately in its normal meaning to animals, and can be extended reasonably well to certain other categories, such as plants, but surely not to numbers. As a consequence, we can regard the incoherence in 1.1.3 as a product of a category mistake.

With the concept of a category mistake we can clearly define statements as those declarative sentences for which the property of truth is appropriately defined. And we can state why questions, commands, and exclamations are not statements in virtue of the fact that it would be a category mistake to apply truth and falsity to any of them.

Now given our definition of an argument, 1.1.4 below is not an argument because no statements are made in the sequence.

Ex. 1.1.4

How many paper cups are on the table? Ouch! Close the door.

Having gone through the fundamental reason why arguments always and only involve statements it is time to make a careful qualification about how we actually speak. In many instances we present arguments by using questions. We use questions for purposes of style, presentation, and perhaps even sarcastic flavor. Nevertheless, even though we use questions when presenting an argument, the question is only present in the implicit argument. Were the argument made explicit no question would be present.

Ex. 1.1.5

Can you find a reason to vote for a Republican next term? So, you should vote for a Democrat.

The sequence of statements above contains a question followed by a statement. The argument that is offered above really gets translated out as the following.

If there is no reason to vote for a Republican next term, then you should vote for a Democrat. There is no reason to vote for a Republican next term. So, you should vote for a Democrat.

So, it is important to keep in mind that the technical definition of an argument from a logical point of view requires that all of the elements of it are statements. Keeping this in mind should guide one in extracting from a particular context involving questions, commands, or exclamations whether an argument might be present implicitly. One should not conclude immediately from the fact that questions, commands, and exclamations are being used in a passage that the author of the passage does not intend to be offering an argument. Rather, one should critically think about what is being said in order to determine whether an argument is implicitly available under some reasonable interpretation.

1.2 Keys for identifying arguments

Arguments occur in a variety of contexts, and one of the most common contexts they occur in is the natural language of the speaker making the argument. For example, native English speakers typically present arguments in English, while native German speakers typically present arguments in German. In addition, an argument is offered only when a speaker intends for a set of propositions to support or prove a proposition. This intention is often marked both in speech and writing by the use of inference indicators. *Inference indicators* are words or phrases that are used to signal the presence of an argument. We use inference indicators to help others recognize when we are giving an argument. There are two kinds of inference indicators. *Conclusion indicators* signal that the sentence that contains them or to which they are prefixed is a conclusion from previously stated premises. *Premise indicators* signal that the sentence to which they are prefixed is a premise. Below is a list of typical conclusion and premise indicators. The list is not exhaustive.

Premise Indicators	***Conclusion Indicators***
For	Therefore
Since	Thus
Because	Hence
Seeing that	Assuming that
Granted that	For this reason
This is true because	Accordingly
The reason is that	Consequently
For the reason that	This being so
In view of the fact that	It follows that
It is a fact that	The moral is
As shown by the fact that	Which proves that
Given that	Which means that
In as much as	From which we can infer
One cannot doubt that	So

Premise and conclusion indicators are the main clues for identifying arguments and analyzing their structure. When placed between two sentences to form a compound sentence, a conclusion indicator signals that the first sentence expresses a premise and the second a conclusion from that premise. A premise indicator placed between two sentences to form a compound sentence signals that the first sentence is a conclusion, and that the second sentence is a premise.

Ex. 1.2.1

There is no milk in the house, so I need to go to the market.

The conclusion indicator ‘so’ signals that ‘There is no milk in the house’ is a premise in support of ‘I need to go to the market’.

There is a fire on the mountain, since there is smoke over there.

The premise indicator ‘since’ signals that ‘there is a fire on the mountain’ is a conclusion that is being drawn from ‘there is smoke over there’.

In addition to being aware of premise and conclusion indicators, it is important to the process of identifying arguments that one recognizes the difference between an argument, an explanation, and an illustration. One reason for this is that some premise indicators and conclusion indicators are contextual. On some occasions of use ‘because’, ‘since’, ‘for’, ‘thus’ are used as part of an explanation or an illustration, and not to indicate that a premise is present or that a conclusion is being drawn.

An *illustration* is a sequence of statements in which examples and instances occur along with other statements, and where the statements combined together exemplify a single statement.

Ex. 1.2.2

In spite of our knowledge that apes and monkeys are much like
us in terms of self-awareness, intelligence, and importance of
social bonds, we have yet to come to grips with the moral implications of this knowledge. Thus, we still subject them to behavior that, were a human the victim, could only be described as torture.

In the passage above the conclusion indicator ‘thus’ is present, which strictly speaking should signal that the statement that comes after it is being concluded on the basis of the statements before it. However, the passage is only an illustration of the claim 'we have yet to come to grips with the moral implications of this knowledge'. The illustration of our failure to come to grips with the moral implications of this knowledge is made through the claim that we subject them to behavior that simply is torture.

An *explanation* is a sequence of statements that purports to shed light on some event or phenomenon that is usually accepted as a fact.

A good way to determine whether a passage contains an argument or an explanation is to use the *why-that test*:

Is the passage intended to show that something is the case or show why something is the case?

If the passage is intended to show that something is the case, it is an argument. Arguments are sequences of statements intending to establish a single statement. If the passage is intended to show why something is the case, it is an explanation. Explanations are sequences of statements that purport to show why a single statement is the case.

Ex. 1.2.3

The Challenger spacecraft exploded after liftoff because of an O-ring failure in one of the booster rockets prior to SRB1 separation.

'Because' is typically a premise indicator. However, in the context above it is only playing the role of an explanation indicator. The passage does not attempt to establish the fact that the Challenger spacecraft exploded. Rather, it attempts to provide an explanation of why it exploded.

A *report* is a passage in which one person states what another person said. Reports occur in all sorts of contexts, but one common context is the news.

Ex. 1.2.4

Jane Smith reports that Frank Johnson has argued that the Fed should cut tax rates, since tax rate cuts would aid the economic stimulus plan.

In this passage, Frank Johnson surely has argued that the Fed should cut tax rates based on the premise that tax rate cuts would aid the economic stimulus plan. However, Jane has not argued this, nor has Jane endorsed the premises or the conclusion of the argument. She has simply reported the argument.

It is important when looking for arguments both when people are talking and in writing that one pay attention to clues that signal that an argument is being presented rather than an illustration, explanation, or report.

Premise and conclusion indicators often signal that an argument is present. However, one must also pay attention to what kind of passage one is reading.

1.3 Atomic and compound arguments

Arguments are like chemical compounds, they have parts; there are basic units, and more complex units that can be put together to make further arguments. *Atomic* arguments have only one conclusion. *Compound* arguments have more than one conclusion. In an atomic argument all premises are intended to jointly establish a single conclusion. In a compound argument premises are offered in support of one conclusion, from which other premises are then additionally offered to support another conclusion. In compound arguments premises that are stated in support of one conclusion may not support another conclusion. Compound arguments are said to have a *main conclusion* and at least one *sub-conclusion*. A sub-conclusion of a compound argument is a conclusion that is reached by the premises of the argument, and then is used along with other premises to reach the main conclusion.

Ex. 1.3.1

Martinsville is in Kansas. Every town in Texas has at least 50 people living in it. Martinsville is a town. Martinsville has 20 people living in it. So, Martinsville is not in Texas. If Martinsville is not in Texas, then Martinsville is in Kansas.

In example 1.3.1 the sub-conclusion is that Martinsville is not in Texas. It is reached on the basis of the second, third, and fourth sentence preceding it. The main conclusion is the first sentence that Martinsville is in Kansas. The whole argument can be viewed in the following way:

Every town in Texas has at least 50 people living in it.
Martinsville is a town.
Martinsville has 20 people living in it.
∴ Martinsville is not in Texas.
If Martinsville is not in Texas, then Martinsville is in Kansas.

Martinsville is in Kansas.

‘∴’ is a formal symbol marking that the sentence following it has been drawn from the sentences above it. ‘____________’ is a formal symbol marking that the sentence underneath it is the main conclusion drawn from the sentences above it coming after the ‘∴’. In general, we put arguments into *standard form*. In standard form premises come before the conclusion, and the conclusion is marked by either ‘∴’ or a ‘____________’ above the conclusion.

1.4 Kinds of arguments

Perhaps one of the central projects of logic and critical reasoning is providing a classification of different kinds of arguments of and establishing evaluative criteria for the different kinds.

Ultimately, when an argument is presented what one wants to know is whether the argument is good. Remember, the point of an argument is to offer reasons for the truth of a proposition. The premises encode the reasons, and the conclusion is the proposition supported by the reasons. The natural question raised by any argument is, “should one accept the conclusion given the reasons offered.”

In evaluating arguments the first thing that one must turn to, after determining what the premises and conclusion are, is determining what kind of argument is being offered. Recall from 1.1 that an argument makes two kinds of claims. A factual claim, that the premises are true; and an inferential claim, that the premises support the conclusion. The notion of *support* is where we turn to determine what kind of argument is being offered. From a logical point of view there are basically two kinds of support that can be offered.

Inductive support – an argument is inductive when the premises make the conclusion likely to be true. In particular, the premises make the conclusion more likely to be true than false, but the premises do not guarantee the conclusion.

In an inductive argument an author intends it to be the case that even if all of the premises of the argument are true it is possible that the conclusion is false. The premises of an inductive argument make the conclusion probable. They only guarantee the conclusion in a limiting case.

Ex. 1.4.1

It has been rainy all month long. So, it will probably be rainy again tomorrow.

Deductive support – an argument is deductive when the premises guarantee the conclusion.

In a deductive argument an author intends it to be the case that if the premises are true; the conclusion has to be true.

Ex. 1.4.2

Either a Republican candidate won the election or a Democratic candidate won the election. A Republican candidate did not win the election. Therefore, a Democratic candidate won the election.

Although both kinds of argument are equally important, our concentration will be on deductive arguments.

Chapter 2
Diagramming Arguments

2.1 Elements of diagrams

Once an argument has been identified, it is often useful to diagram its structure. In this section we will look over techniques for diagramming arguments.

An argument diagram involves the application of three kinds of symbols.

Numbers: 1, 2, 3, 4, 5, 6, …

Joint: +

Supports: ________

Numbers are used as names of the claims that constitute the argument, such as the premises and the conclusion. Placing a '+' between two, or more, premises means that the premise are jointly working together to support a conclusion. The line is used to identify a support relation between premises and a conclusion. The premises go above the line, and the conclusion underneath.

Consider the following arguments:

Example 2.1.1

Some people that attend church don't believe in God. But some people that don't attend church believe in God. So, whether one belives in God or not cannot be determined by whether they go to church.

[(1) Some people that attend church don't believe in God]. But [(2) some people that don't attend church believe in God]. So, [(3) whether one believes in God or not cannot be determined by whether they go to Church.]

$$\frac{(1) + (2)}{(3)}$$

Example 2.1.2

All good athletes eat healthy food. Since, all good athletes train hard. And healthy food is necessary for training hard.

[(1) All good athletes eat healthy food.] Since, [(2) all good athletes train hard.] [(3) And healthy food is necessary for training hard.]

$$\frac{(2) + (3)}{(1)}$$

While in example (2.1.1) we have a straightforward argument that comes with premises first, and conclusion last, in example (2.1.2) we have conclusion first, and premise last. In both cases the premises are working together to support the conclusion.

Example 2.1.3

Since half way through the war we have all been tired of the fact that the economy is not doing well. So, it is time for a change. Given that it is a time for change, we ought to vote for the presidential candidate that wants to change things the most. So, we can't vote for Jim Brown. So, we should vote for Mary Moore.

Since [(1) half way through the war we have all been tired of the fact that the economy is not doing well.] So, [(2) it is time for a change.] [(3) If it is a time for change, we ought to vote for the presidential candidate that wants to change things the most.] [(4) Mary Moore wants more change than Jim Brown.] So, [(5) we should vote for Mary Moore.]

$$\frac{\frac{(1)}{(2)} + (3) + (4)}{(5)}$$

In example (2.1.3) we are dealing with a complex argument that has two atomic arguments. In the first argument premise (1) is used alone to support (2), which with (3) and (4) is being offered in support of (5).

Example 2.1.4

Lentils are healthy. There are many reasons for this. They reduce heart disease. They are a great source of fiber and protein. They are low in fat and simple carbohydrates.

[(1) Lentils are healthy.] [(2) There are many reasons for this.] [(3) They reduce heart disease.] [(4) They are a great source of fiber and protein.] [(5) They are low in fat and simple carbohydrates.]

$$\frac{(2)\quad (3)\quad (4)\quad (5)}{(1)}$$

In this argument the conclusion is the first statement, that lentils are good. Each statement after that provides an independent argument for the conclusion. Example (2.1.4) should **not** be diagrammed as follows.

$$\frac{(2) + (3) + (4) + (5)}{(1)}$$

This diagram suggests that premises together support the conclusion, but that no premise *alone* supports the conclusion. The second argument is weaker than the first. In the second the conclusion is true only if all of the premises are true. In the first argument the conclusion is said to be true just in case one of the premises is true. We might look at the first argument as follows as well.

$$\frac{(2)}{(1)} \qquad \frac{(3)}{(1)} \qquad \frac{(4)}{(1)} \qquad \frac{(5)}{(1)}$$

to make the point clearer.

Example 2.1.5

If the Sharks aren't strong enough to beat the Flyers, they won't be able to beat the Red Wings. From last week's performance they don't look strong enough to beat the Flyers. So, it is likely they won't beat the Red Wings.

[(1) If the Sharks aren't strong enough to beat the Flyers, they won't be able to beat the Red Wings.] [(2) From last week's performance they don't look strong enough to beat the Flyers.] [(3) So, it is likely they won't beat the Red Wings.]

$$\frac{(1) + (2)}{(3)}$$

Example 2.1.6

If Jackson were president, he'd have lowered taxes by now. If taxes were lowered, we'd have more money. If we had more money we'd be happier. If Jackson were president we'd be happier.

[(1) If Jackson were president, he'd have lowered taxes by now.] [(2) If taxes were lowered, we'd have more money.] [(3) If we had more money we'd be happier.] [(4) If Jackson were president we'd be happier.]

$$\frac{(1) + (2) + (3)}{(4)}$$

Example 2.1.7

We shouldn't vote for Jackson. Jackson is too radical. Jackson is too inexperienced. Jackson's lack of experience would make him a dangerous president. One shouldn't vote for anybody who is too radical. One shouldn't vote for anybody who would be a dangerous president.

[(1) We shouldn't vote for Jackson. [(2) Jackson is too radical.] [(3) Jackson is too inexperienced.] [(4) Jackson's lack of experience would make him a dangerous president.] [(5) One shouldn't vote for anybody who is too radical.] [(6) One shouldn't vote for anybody who would be a dangerous president.]

$$\frac{(2) + (5)}{(1)} \qquad\qquad \frac{(3) + (4) + (6)}{(1)}$$

In this example we have two independent arguments for the conclusion. Both arguments require joint claims. The first argument joins (2) and (5), and the second argument joins (3), (4), and (6).

Two general structures for complex arguments:

A complex argument has an *independent* structure when there are at least two distinct arguments for the main conclusion.

A complex argument has a *dependent* structure when all premises jointly support the conclusion.

Chapter 3
Fallacious Reasoning in Argumentation

3.0 Fallacies in General

Fallacious reasoning is bad reasoning, reasoning that is to be avoided in constructing an argument; and to be detected, pointed out, and rejected when present. There is no straightforward theory of what constitutes fallacious reasoning. Fallacious reasoning is simply reasoning that is defective in some way, and may even be defective in a way that is deceptive, so as to seem as if it is good reasoning when it actually is not. Although there is no complete theory of fallacious reasoning, there are several cases that, since the time of Aristotle, have been agreed to involve defective reasoning. In general, logicians distinguish kinds of fallacious reasoning by attending to a fundamental distinction:

Formal: an argument is said to be formally fallacious when the argument involves a bad pattern of reasoning independent of the content and context of the argument. Formally fallacious arguments are bad because the abstract pattern and structure of reasoning the argument is an instance of is itself one that does not preserve truth or increase the probability of the conclusion. Formal fallacies can be detected merely by inspecting the form of the argument. All invalid arguments are formally fallacious.

Informal Fallacies: an argument is said to be informally fallacious when form, context, and content all play a role in determining that the reasoning involved is bad. Informal fallacies require attention to be paid to the language used and the context of the dialectic in which the argument occurs.

Consider the following argument:

(D)

	D1.	All factories are plants.
	D2.	All plants are things that contain chlorophyll.
∴	D3.	All factories are things that contain chlorophyll.

If one were not to pay attention to the content of what is being said, one would be deluded into thinking the argument is good because it *seems* like an instance of a very good pattern of reasoning:

(E)

	E1.	All F are P.
	E2.	All P are C.
∴	E3.	All F are C.

However, if we inspect the content of what is being said, the form is actually quite different. The word ‘Plant’ is being used in two different senses. In D1 we are not talking about a living organism. In D2 we are talking about a living organism. As a consequence, we cannot say the form of the argument is (E). The real form of the argument is.

(F)

	F1.	All F are P.
	F2.	All K are C.
∴	F3.	All F are C.

However, (F) is not a good form of reasoning. So (D) is fallacious. The reason why (D) is fallacious is because it involves lexical equivocation, using one syntactic string under two different meanings. Later we will turn to this fallacy when we consider the class of semantic fallacies.

The focus of the study of fallacies in the remaining notes will be on informal fallacies. Informal fallacies can be divided up into several different types depending on the central factor. The fallacy types to be discussed are: *fallacies of relevance, fallacies of presumption, fallacies of weak induction, fallacies of ambiguity, and fallacies of grammatical analogy*. Because informal fallacies require attention to content and context, in the explanations to follow both fallacious and non-fallacious instances of the fallacies’ form will be pointed out when necessary.

3.1 Fallacies of Relevance

An argument is guilty of committing a *fallacy of relevance* when the premises of the argument are not relevant to the conclusion of the argument. Arguments that commit the fallacy of relevance often also commit other fallacies both formal and informal.

(1) Appeal to Force:

An argument commits the fallacy of appeal to force when the conclusion is supported by a threat.

Note that the threat may be either implicit or explicit. Contrast the following two cases, the first an explicit case, the second an implicit case.

Examples:

You should give me your next paycheck. If you don't I will kill your dog.

You should let me borrow your Porsche tonight. You wouldn't want to come home to find your precious children missing.

No Fallacy:

You should attend logic class on a regular basis. Failure to do so could result in a bad grade.

(2) Appeal to Pity:

An argument commits the fallacy of appeal to pity when the conclusion is supported by evoking pity on the part of the audience.

Example:

You should give me a better grade for the course. I had a really rough semester. My grandparents died and my parents got divorced.

A word of caution:

Not every argument that evokes pity on the part of the audience is guilty of committing the fallacy of appeal to pity.

Example:

People who train dogs to fight torture them to make them become more aggressive, force them to exercise to the point of exhaustion every day to become stronger, starve them before the fight, subject them to fights that often times results in their death or permanent injury, and then usually abandon them or put them down when they are past fighting age. This practice should be forbidden.

(3) Appeal to the people:

An argument commits the fallacy of appeal to the people when the arguer relies on the audience's desires to be loved, esteemed, admired, valued, recognized, and accepted by others in order to get them to accept the conclusion.

There are two approaches involved in the appeal to the people fallacy (distinguished by the role of "the people"):

Indirect -- when the audience is a single individual
Direct -- when the audience is a group of individuals

Three forms of the indirect approach:

(a) The bandwagon argument:

An argument commits the bandwagon form of the appeal to the people fallacy when the reason given for accepting the conclusion is that many others accept it.

Example:

Toyota makes better cars than Lamborghini. Just look at their annual sales. Way more people buy Toyotas than Lamborghinis.

(b) The appeal to vanity:

An argument commits the appeal to vanity form of the appeal to vanity fallacy when the arguer associates the conclusion with some preferred way of being.

Example:

Men who drink protein shakes have hot bodies and get all the women. So, if you're a man, you should drink protein shakes.

(c) The appeal to snobbery:

An argument commits the appeal to snobbery form of the appeal to the people fallacy when the arguer associates the conclusion with some elite group of people.

Example:

The best way to spend your leisure time is to play golf. That's how all of society's elite spend their leisure time.

(4) Argument against the person (*ad Hominem*):

An argument commits the fallacy of argument against the person when the arguer rejects another person's statement/argument *due to the nature* of the person who made the statement/argument rather than on the merits of the statement/argument itself.

There are three main types of *ad hominem* arguments:

(a) *ad hominem* abusive:

An argument commits the abusive form of the *ad hominem* fallacy when the arguer responds to another person's argument by verbally attacking them.

Example:

Our mayor wants to get an ordinance passed that would increase sales tax even more. But our mayor is so incompetent. He just messes everything up because he's an idiot. So, we shouldn't support him in this ordinance.

(b) *ad hominem* circumstantial:

An argument commits the circumstantial form of the *ad hominem* fallacy when the arguer rejects another person's statement/argument on the grounds of some *circumstances* surrounding them.

Example:

Bob thinks that war is wrong. But he has never been in a war, so how can he know? He must be wrong about that.

(c) *ad hominem tu quoque* ("you too") fallacy:

An argument commits the *tu quoque* form of the *ad hominem* fallacy when the arguer rejects another person's statement/argument on the grounds that that person is hypocritical or arguing in bad faith.

Example:

Mary told me that she thinks that people shouldn't smoke because it's bad for one's health, it smells bad, and it is expensive. But I've seen her smoking before, so obviously she's wrong about smoking being bad.

Important:

Not all instances of attacking the character/credibility of an individual is an instance of the *ad hominem* fallacy.

Example:

Bob is a mean jerk. He hits people, calls people names and steals money from people.

(5) Accident:

An argument commits the fallacy of accident when a general rule or law is applied to a specific case that it was not intended to cover. Since the general rule is not meant to cover the kind of case in question, the rule is irrelevant.

Example:

Plunging a knife into another person's chest is illegal. Given that surgeons do precisely this when operating, it follows that surgeons should be arrested.

(6) Straw Man:

An argument commits the straw man fallacy when the arguer distorts another person's *actual* position or argument and attacks the (weaker) misrepresentation.

Example:

Republicans are generally opposed to abortion because they value the inherent sanctity of human life. But it would be wrong, as they seem to be arguing, to be opposed to all forms of population control. The current human population on Earth is a serious issue that needs to be addressed.

Example:

John says he is opposed to violence. Apparently, John endorses a view in which he doesn't even think it's okay to try to physically stop someone from attacking him and his family. His position seems absurd.

(7) Missing the Point:

An argument commits the fallacy of missing the point when the premises of an argument support one conclusion, but the author of the argument instead arrives at another, irrelevant conclusion.

Example:

Every time Melissa has peanuts with her lunch, she has an incredibly severe allergic reaction. She should probably start having peanuts with dinner instead.

Important:

All fallacies of relevance in some sense *miss the point*. However, the fallacy of missing the point is not simply that the author has drawn an irrelevant conclusion it is that the author has draw a conclusion that is not the one intended and it is obvious what the intended conclusion is. If an argument commits the fallacy of missing the point then it is relatively easy, as in the case above, to state the intended conclusion. The conclusion one probably *should* draw from the stated premises would be something like: "Therefore, Bob should probably not drink as much anymore."

(8) Red Herring:

An argument commits the red herring fallacy when the arguer changes the subject mid-argument to draw the attention of the audience away from the original target. The new subject to which the arguer diverts the argument is generally related to the original topic so as to obscure the shift.

Example:

Abortion is wrong because it puts an end to a life that is inherently valuable. And it's not just human lives that are valuable. Dogs, bunnies and cats are precious creatures. Just look at them. They are so cute. It would be very wrong indeed to hurt these little animals.

Important:

There is a certain similarity between the *straw man* fallacy and the *red herring* fallacy. Because both fallacies involve some new position or topic, which is irrelevant to the topic of interest, they are easily confused.

To distinguish the two fallacies, note the role that the irrelevant position/topic plays in them.

In the *straw man* fallacy the new position is presented *as someone else's* so that the arguer can *attack an easier position*.

In the *red herring* fallacy the new topic is brought up so that the arguer may decisively "win" an argument on *that* topic and thereby convince the audience *without engaging the original issue*.

Examples:

It would be very devastating to our economy if we raised taxes right now. My neighbor is very hardworking, yet poor and cannot afford to pay for any additional expenses right now without losing his home. He has such a good work ethic, though. No citizen who works that hard should ever be evicted.

George has put forward the view that taxes should be raised. But if we were to raise taxes, as he suggests, we would basically be saying that it is okay to take away almost every dollar that we earn and give it to the government. This would mean that hardworking people would be put out on the streets. Since it would be a mistake to allow this, George must be mistaken.

The first of these arguments commits the red herring fallacy. The arguer means to establish that taxes should not be increased. Instead of arguing about taxes, however, the arguer argues about how it is wrong to evict people with good work ethics (a related, but irrelevant topic).

The second argument is very similar but commits the straw man fallacy. George's position is distorted (from being a defense of not raising taxes to being a defense of not evicting hardworking people), and the arguer attacks this distorted position.

3.2 Fallacies of Weak Induction

An argument is guilty of committing a *fallacy of weak induction* when the premises of the argument (although *relevant* to the conclusion) are not strong enough to support the conclusion.

(1) Appeal to unqualified authority:

An argument commits the fallacy of appeal to unqualified authority when the conclusion of the argument is supported by the claims of someone who is not trustworthy for some reason.

Example:

Carlos Prieto, the famous Mexican concert cellist, said in an interview that there was a recent discovery of a new galaxy near the Andromeda Galaxy. He must be right about that.

In this example, the authority is unqualified because he lacks expertise in the field of interest. An authority may also be unqualified when that person has some "interest" in the conclusion.

Example:

Jason Anderson, the president of Quality Paperback Books Inc. has argued that devices which allow you to purchase and read books electronically should be banned, since it hinders the sales of hardcopy books which have been very important to our civilization for so many thousands of years.

Note that in this case the authority is unqualified because of a *circumstance* of that authority (namely, that he is the president of a company that can only survive if the sales of hardcopy books continues).

It may seem, then, that our claiming the above argument to commit the fallacy of appeal to unqualified authority *itself* commits the fallacy of *ad hominem* circumstantial. Although it is in the same neighborhood as *ad hominem*, evaluating someone's argument as fallacious in this way is not itself fallacious. What would be fallacious is the following:

Jason Anderson, the president of Quality Paperback Books Inc. has argued that devices which allow you to purchase and read books electronically should be banned, since it hinders the sales of hardcopy books which have been very important to our civilization for so many thousands of years. Given that he has a vested interest in the company, we should ignore his arguments.

(2) Appeal to Ignorance:

An argument commits the fallacy of appeal to ignorance when the conclusion of an argument makes a definite assertion about something *on the grounds* (premises) that nothing has been proven one way or another about that thing.

Example:

Archaeologists have not yet found the missing link between the genus Australopithecus and Homo. So, there must not be one.

No Fallacy:

I have never seen, from my house in San Francisco, the sun rise from the east. So, it probably never will in my lifetime.

(3) Hasty Generalization:

An argument commits the fallacy of hasty generalization when a statement about *a group* is supported by an unrepresentative sample of that group.

Example:

The last two times we ate at that taco shop, the tacos weren't very good. They probably have never made good tacos.

My last two girlfriends complained a lot. It has become very apparent to me from these experiences that all women are just complainers.

(4) False Cause:

An argument commits the fallacy of false cause when the link between the premises and the conclusion depends upon a causal connection that does not exist. There are three types of false cause arguments.

(a) *post hoc ergo propter hoc* -- after this, therefore on account of this

An argument commits the *post hoc (ergo propter hoc)* form of the false cause fallacy when one event is taken to be the cause of another simply because the "effect" occurred *after* the "cause".

Example:

Right after the president's inaugural speech, it started raining. The president must have made it rain somehow.

(b) *non causa pro causa* -- not the cause for the cause

An argument commits the *non causa pro causa* form of the false cause fallacy when there *is* a causal relationship between two things (as the argument indicates), but the argument confuses the cause for the effect.

Example:

Every time there is a thunderstorm, the power goes out across the entire town. Our power shortages must be somehow causing these storms to happen.

(c) oversimplified cause:

An argument commits the oversimplified cause form of the false cause fallacy when there are in fact *many* causes for an event, but the argument indicates that there is just one.

Example:

In the years following our new governor stepping into office, crime rates went way up. It must be all our governor's fault that these crime rates went up.

(5) Slippery Slope:

An argument commits the fallacy of slippery slope when the conclusion is supported by a dubious chain of causes.

Example:

It would be a very bad idea to legalize homosexual marriage. Although homosexual marriage itself is not wrong, if men were allowed to marry men and women were allowed to marry women, then pretty soon people would be wanting to marry animals and then probably bizarre things like furniture and potting soil. And if this happened, then this would ruin the entire point of marriage for heterosexuals in the first place – their once meaningful relationship with their spouse will become pointless. We shouldn't allow homosexual marriage.

(6) Weak Analogy:

An argument commits the fallacy of weak analogy when the conclusion is supported by an analogy between two things which (although similar) are relevantly different.

Example:

I can drink a gallon of water in an hour without any negative health consequences. Since whisky is similar to water in many ways, I could probably also drink a gallon of it without any serious consequences as well.

A brief aside about analogical arguments:

Analogical arguments (or, arguments by analogy) are inductive arguments of the following general form:

(A)

A1. *x* is *F*.
A2. *y* is similar (in some regard) to *x*.
∴ A3. *y* is also *F*.

An analogical argument is strong when the similarity claimed to hold between *x* and *y* (A2) is sufficient to warrant us in inferring that *y* is *F* (A3) from the fact that *x* is *F* (A1).

Examples:

My dog can eat about a pound of dog food per day. My neighbor's dog is the same exact age and breed as my dog and came from the same litter. So, my neighbor's dog can also eat about a pound of dog food per day.

My dog can eat about a pound of food per day. My neighbor's dog has the same fur color as my dog. So, my neighbor's dog can also eat about a pound of food per day.

3.3 Fallacies of Presumption

An argument commits a fallacy of presumption when the premises of the argument assume or *presuppose* that of which they are offered in support.

(1) Begging the Question:

An argument commits the fallacy of begging the question when the arguer creates the illusion that inadequate premises provide adequate support for the conclusion by either. There are three basic types of begging the question fallacy:

(a) Leaving out a key premise

(b) Restating the conclusion as a premise
(c) Arguing in a circle.

(a) Missing (controversial) premise:

Example:

Lowering taxes would cause the crime rate to absolutely plummet to never-before seen low levels. Obviously, we should lower taxes.

(b) Restating the conclusion as a premise:

Example:

People like to drive Fords, because Fords are the vehicles that people really like.

(c) Circular reasoning:

Example:

I know that all of my clear and distinct perceptions are true. I know this because there is a non-deceptive God who would not allow me to be mistaken in this way. I know that there is a non-deceptive God because I clearly and distinctly perceive that there is such a God.

(2) Complex Question:

The fallacy of complex question occurs when a person is asked one question that actually contains two (or more) separate questions. Although the question (as a non-statement) is not an argument, it involves an implicit argument (usually for the condition presumed).

Example:

Hey, Al, have you stopped growing marijuana yet?

Explanation:

If the answer is 'yes', then Al admits that he was growing marijuana in the past. If the answer is 'no', then Al admits that he is currently growing marijuana.

Example:

Does your boss know that you have stolen from the company?

(3) False Dichotomy:

An argument commits the fallacy of false dichotomy when a premise contains a choice between two options that are implied to be jointly exhaustive but which are not.

Such a choice is generally expressed by some *either…or…* statement.

Example:

People are either fat or skinny. James is definitely not fat. So he must be skinny.

Example:

The cultural diversity of the United States can be represented by one of the following metaphors: a melting pot (a place where many races and ethnicities come and mix together) or a salad bowl (a place where many races and ethnicities come but remain separate and isolated from each other). Well, it certainly is not a melting pot, so it must be a salad bowl.

Explanation:

While it might be true that the various races and ethnicities that reside in the United States have not mixed and intermarried to the point where there is but one distinctive new kind of ethnicity that is the product of a mixture of the originally separate peoples, this does not imply that the United States is therefore only a collection of completely separate peoples who do not mix or intermarry at all. Rather, what is actually true of the cultural diversity in the United States is that there is some mixing of various ethnicities and there is also some remaining separate. It doesn't have to be just one or the other.

(4) Suppressed Evidence:

An argument commits the fallacy of suppressed evidence when the conclusion is supported by the premises offered, but at least one vital fact that would undermine this support is left out.

Example:

This class meets on Mondays, Wednesdays, and Fridays. February 17, 2094 is on a Wednesday, so I expect to see you all then.

Explanation:

While it is true that February 17, 2094 is a Wednesday, we obviously will not be attending this school in that many years from now. That it was not stated in the first premise which year it was, is what renders the argument fallacious. So, the presented claims are undermined by an obviously relevant piece of information, is the particular Wednesday in question is in 2094 and not the present year in which you are enrolled in this class.

3.4 Fallacies of Ambiguity

An argument commits a fallacy of ambiguity when either a term or statement in the argument is unclear (or ambiguous), and this ambiguity obscures some bad reasoning.

(1) Equivocation:

An argument commits the fallacy of equivocation when a key term in the argument is used in two different senses.

Examples:

A small piece of paper is light. Something cannot be both light and dark at the same time. Therefore, a small piece of paper cannot be dark.

A hamster is an animal. Therefore, a large hamster is a large animal.

Note that in the second of these arguments the ambiguous term is 'large'. The ambiguity of this term turns on the fact that the meaning of a given use of it depends upon the *context of utterance*. In order to identify equivocation, look for terms that either have numerous meaning, or have meanings that vary by context.

(2) Amphiboly:

An argument commits the fallacy of amphiboly when the conclusion depends upon the misinterpretation of a grammatically ambiguous statement.

Example:

Jeremy told me that he just read a short story about a jewel heist in the Louvre in Paris. But I know that Jeremy has never been to Paris. So, he must be lying about reading that novel there.

Examples of grammatically ambiguous statements:

I saw a TV show about an earthquake last month.
Jim got very angry at his mother's house.

3.5 Fallacies of Grammatical Analogy:

An argument commits a fallacy of grammatical analogy when it is structurally analogous to a good argument, but in which an unwarranted inference is made.

(1) Composition:

An argument commits the fallacy of composition when it is claimed that a *whole* must have a property on the grounds that the *parts* have that property.

Examples:

None of the books in this stack weighs over 3 pounds. Therefore, this stack of books does not weigh over three pounds.

No fallacy:

Since every piece of the table is made out of wood, the table is made out of wood.

(2) Division:

An argument commits the fallacy of division when it is claimed that the *parts* (of a whole) must have a property on the grounds that the *whole* has that property.

Example:

This house is very big. So, every room in the house must be very big.

No fallacy:

Jake's car is in Los Angeles. So, every part of his car is in Los Angeles.

In general, when looking at potential cases of the composition and division fallacy, be sure to pay attention to the property that is at issue. Certain properties transfer from *whole to part*, and other transfer from *part to whole*.

3.6 Additional fallacies

(1) Genetic Fallacy

An argument commits the genetic fallacy when the arguer infers that one thing has a property because it originated from something that had the property.

Example:

Both of Chris's parents are alcoholics. So, Chris must be an alcoholic as well.

No Fallacy:

This chair must be made out of wood, since it was made only from that oak tree.

Important:

The water in the Monterey Bay Aquarium is saltwater because every drop of it came from the Pacific Ocean, which is a saltwater ocean.

(2) Gambler's Fallacy

An argument commits the gambler's fallacy when one believes that an event will occur because in the past it has not occurred, and the events are independent of one another.

Example:

The last time someone used this slot machine, they hit a jackpot. So, the next time someone uses that same slot machine, it definitely will not be a jackpot.

3.7 Some commonly confused fallacies

1. Straw Man vs. Red Herring

Often Straw Man and Red Herring are hard to separate. The key thing to focus on is whether the author intends

(a) to introduce an argument that is irrelevant because it is easier to defeat.
(b) to introduce an irrelevant topic in order to distract attention from the main point.

Case (a) is Straw Man. Case (b) is Red Herring.

2. Hasty Generalization vs. Composition

Hasty Generalization and Composition share in common the feature that the arguer moves from attributing a property on a "small scale" in the premises to attributing a property on a "large scale" in the conclusion. This feature of moving from a small scale to a large scale is what makes the two hard to separate. The key is to focus on whether the author intends

(a) to take a property that is had by a sub-group of a total population and *generalize* to the whole population.
(b) to take a property that is had by *every part* of a whole, and maintain that it is had by the whole.

Examples:

300 out of the 37,000 San Jose State University students live on campus. So, more than 60 percent of San Jose State University students must live on campus.

Every part of the watch is small. So, the watch is small.

The first argument is a case of Hasty Generalization. Surely, a sample of 300 students out of 37,000 is far too few to generalize to the conclusion that more than 60 percent of students live on campus.

The second argument is a case of composition. Size is the kind of property that aggregates. Independent parts can be small, while coming together to form a unit that is large.

3.8 Techniques for identifying fallacies

When attempting to identify whether a passage contains a fallacy use the following procedure.

(a) Does the passage contain an argument? If so, what is the conclusion?
(b) Does the passage contain a controversial claim?
(c) Do any of the central claims rely on expertise?
(d) Is a set of options assumed to be exhaustive?
(e) Do any words appear to be used in different ways?

Keep in mind that a passage may contain more than one fallacy. Keep in mind that often times by asking a series of more precise questions one can arrive at the precise fallacy in question. For example, in reading an argument you might feel that the premises are irrelevant to the conclusion. But once you ask "why?" you may discover it is because of a Strawman argument is present?

Take note of different readings of the passage. Often a fallacy will be revealed when the context in which the argument is being given is focused on.

Be sensitive to common problems in passage types.

The most important thing about identifying fallacies is not the name of the fallacy, but a characterization and understanding of what the problem is. The capacity to articulate the error and defend it as problematic is more important than being able to cite the name, but not explain the problem or defend that the argument commits the fallacy in question.

Unit 2
Formal Evaluation of Arguments

Chapter 4
Criteria for Evaluating Arguments

4.1 The form-content distinction

One of the most important distinctions in logic, and an essential component of effective critical reasoning, is the form-content distinction. In order to get a handle on this distinction, consider the following arguments:

(a) If taxes increase, then inflation will increase.
Taxes will increase.
∴ Inflation will increase.

(b) If wages fall, then taxes will increase.
Wages will fall.
∴ Taxes will increase.

(c) All dogs are mice.
All mice are fish.
∴ All dogs are fish.

(d) All birds are animals.
All animals are mammals.
∴ All birds are mammals.

Notice that although (a) and (b), and (c) and (d) differ in content, (a) and (b) share a pattern of reasoning, and (c) and (d) share a pattern of reasoning. (a) and (b) differ in content because (a) is about taxes and inflation, and (b) is about wages and taxes; (c) and (d) differ in content because (c) is about dogs, mice, and fish; while (d) is about birds, animals, and mammals.

The common form that (a) and (b) share is

If P, *then* Q.
P.
∴ Q.

The common form that (c) and (d) share is

All A are B.
All B are C.
∴ *All* A are C.

Isolating and holding constant specific logical phrases in the statements that make up the argument, and abstracting away the content of the non-logical phrases generate the logical form of an argument. Logical form is a consequence of holding some phrases constant, the logical phrases, and abstracting away from the content of the non-logical phrases.

From a logical point of view (a) and (b) are **propositional arguments**; the logical phrases that are relevant to propositional arguments are the logical phrases that connect propositions or sentences together. The propositional connective in (a) and (b) is 'if….then'. It is a connective that joins propositions, in the case of (a) the propositions joined are 'taxes increase' and 'inflation will increase'. Other propositional connectives and operators are 'not', 'or', 'and', 'if and only if'. These connectives and operators will be the focus of study when we turn to propositional logic. In propositional logic, propositions are abstracted away from, and propositional connectives are preserved.

From a logical point of view (c) and (d) are **quantificational arguments**, the logical phrases that are relevant are 'all', 'no', and 'some'. These phrases are known as quantifier phrases because they talk of quantity. They address the question: how many of such and such are so and so? For example, how many birds are animals? Answer: all. Both (c) and (d) are also known more specifically as categorical or Aristotelian syllogisms because they are about quantity relations between category phrases, such as 'birds' and 'animals', and because they have exactly two premises. In order to generate the logical form of a quantificational argument one preserves the quantifier phrases and abstracts away the category or predicate phrases, such as 'dogs', 'fish', and 'students'. Category and predicate phrases are logically irrelevant in quantificational arguments.

The main thing to notice now is that in talking about an argument we talk both about its logical form relative to a holding constant certain logical phrases and abstracting away other phrases and we can talk about its content – the actual statements that make up the argument and which express propositions that are either true or false. The content of an argument has to do with what it is about, the form with the structure of reasoning involved in the argument. Forms of arguments are multiply realizable. They can be instantiated in various particular arguments.

4.2 Validity and Soundness

As noted before, one basic question in critical reasoning concerning a specific argument, and which logic provides us with a partial answer to, is whether the argument is good. Given an argument, our primary goal is to determine whether the argument is good. If the argument is good, then perhaps we should revise our beliefs in light of what the conclusion of the argument is; if the argument is bad, then perhaps we should find reasons to show that it is bad, and in addition not revise our beliefs in light of it.

In assessing whether a deductive argument is good from a logical point of view there are two criteria the argument must satisfy in order to be good. These criteria are necessary conditions, not sufficient conditions on the goodness of a deductive argument. An argument may satisfy these conditions and still be a bad argument (more on how this holds true to follow). However, an argument must satisfy these conditions to be a good argument. The two criteria are the following.

Validity: an argument is valid when it is impossible for the premises to be true, and the conclusion false at the same time.

Soundness: an argument is sound when it is valid and all of the premises are true.

Note: *If an argument is sound, then the conclusion has to be true.*
Here is an argument for this claim using the definition of validity and soundness.

Given that a sound argument is valid, it follows that if the premises are true, the conclusion has to be true; and given that a sound argument has true premises, it follows by the validity of the argument that the conclusion has to be true.

In understanding validity it is important to note that validity is a *formal* property of an argument. Strictly speaking, a particular argument is valid because it is an instance of a valid argument form relative to a scheme of abbreviation. As noted in the last section argument forms are abstract patterns of reasoning that specific arguments share in common. Paying attention to the logically relevant phrases, and abstracting away the logically irrelevant phrases generate argument forms.

In understanding soundness, it is important to note that it is both a *form* and *content*-based property of an argument. It is a formal property of an argument because it includes validity. It is a content property of an argument because the premises have to be true, and in a great many cases the truth or falsity of a premise always depends on the actual content of the propositions expressed.

Let us consider some examples to understand the relation between validity and soundness.

Ex. 4.2.1

(a) If whales are insects, then humans are reptiles.
Whales are insects.
∴ Humans are reptiles.

(b) If humans are mammals, then whales are mammals.
Whales are mammals.
∴ Humans are mammals.

(a) is a case in which the person arguing does not know the facts of biology. However, there is something praiseworthy about the reasoning in (a). If the premises were true, the conclusion would be true. Moreover, (a) is valid. (b) is a case in which the person arguing does know the facts of biology. However, there is something wrong with (b). Even though the premises are true, the conclusion does not follow.

Recall that an argument makes two claims. The factual claim, that the premises are true, and the inferential claim, that the premises support the conclusion. Now, since the arguments above are deductive the inferential claim is the following: if the premises are true, then the conclusion must be true. In (a) the factual claim is false, but the inferential claim is true. In (b) the factual claim is true, and the inferential claim is false. Although the premises and the conclusion are true in (b), the conclusion is not supported by the premises. (a) is a valid, but unsound argument. And because (b) is invalid, it cannot be sound.

(c) If squares are rectangles, then rectangles are shapes.
If rectangles are shapes, then triangles are shapes.
∴ If squares are rectangles, then triangles are shapes.

(c) is a case in which the argument is valid, and sound. It is impossible for the premises to be true and the conclusion false. The argument is an instance of a valid argument form. And in fact, the premises are true.

What is important about validity is that when an argument is valid we know that the reasoning involved is *truth-preserving*. Our reasoning will not move from true premises to a false conclusion. What is important about soundness is that our premises are actually true. When an argument is sound not only is the reasoning good but also the claims that are being made correspond to the way the world is.

However, in evaluating arguments we need to address the following question: Is an argument deemed good simply if it is valid and sound? In the beginning of this section it was noted that validity and soundness are necessary conditions for an argument being good. Now it is time to show why validity and soundness are not sufficient conditions for an argument being good.

(d) All mammals are animals.
∴ All mammals are animals.

An argument is a sequence of statements one of which is supported by the others. (d) is clearly an argument under that definition. The first statement and the second statement are the same, but the first statement is offered in favor of itself. (d) is also a deductive argument. It is impossible for the premises to be true and the conclusion false at the same time (because the premise is the same as the conclusion). So, the argument is valid. And since the premise is true, the conclusion has to be true. So, (d) is a valid and sound argument. However, it is clearly a bad argument. It is a bad argument because it is circular. No reason is offered for the conclusion other than the conclusion. No one who did not already accept the conclusion would accept the conclusion when presented with this argument, since the premise, which is supposed to offer a reason for the conclusion, is the conclusion itself. So, in addition to a good argument being valid and sound, it should be non-circular.

In addition, as discussed above, validity is a formal property of an argument. As a consequence, it is *very important to keep in mind* that an argument is invalid just in case it is possible for the premises to be true and the conclusion false. The fact that the premises are *actually* true, and the conclusion is *actually* true is irrelevant to whether the argument is valid. For example:

Here are some interesting cases to keep in mind.

A valid argument can have false premises and a false conclusion:

All squares are triangles.
All triangles are circles.
∴ All squares are circles.

A valid argument can have false premises and a true conclusion:

All squares are circles.
All circles are rectangles.
∴ All squares are rectangles.

The answer to why this occurs is because validity and invalidity are formal properties of an argument. The reason a valid argument can have false premises and a false conclusion, and false premises and a true conclusion is because the specific argument being made can be an instance of an argument form that is valid.

Both of the cases above are instances of the **categorical argument** form:

All A are B.
All B are C.
∴ All A are C.

This argument form is such that no matter what 'A', 'B', and 'C' stand for the specific argument being made will be valid because any specific argument will be an instance of the valid argument form above (more on this to come in chapters that follow).

Likewise, an argument can have true premises and a true conclusion and be invalid.

All squares are polygons.
All rectangles are polygons.
∴ All squares are rectangles.

In the argument above the premises and the conclusion are true. However, the argument is invalid. The argument is invalid because it is an instance of an invalid argument form. We can show this by determining the arguments logical form, and showing that it has an instance in which the premises are true and the conclusion false.

All A are C.
All B are C.
∴ All A are B.

A logical form is said to be a valid logical form just in case there is no instance of the form in which the premises are true and the conclusion false. As a consequence we can check to see whether the original argument is valid by checking whether it is an instance of a form where the premises could be true and the conclusion false. In addition, if the logical form generated for the argument is the true and complete form of the argument, and it is an invalid form, then the argument itself is an invalid argument.

All cats are animals.
All dogs are animals.
∴ All cats are dogs.

By substituting the category *cat* for A, *dog* for B, and *animal* for C in the argument form a new specific argument is determined. However, this instance has true premises and a false conclusion. This shows that the form above is an invalid form; and thus although the original argument about squares, rectangles, and polygons had true premises and a true conclusion, it is invalid.

So, one should always be careful when talking about the validity of a specific argument to distinguish the actual truth or falsity of the premises and the conclusion in the argument from the possibility of true premises and a false conclusion for the argument form. In addition, actual truth and falsity is often a distraction. When evaluating an argument one should not get locked into the actual truth or falsity of a premise or conclusion. Knowledge of the truth or falsity of a premise is helpful when one is assessing soundness, given that if an argument is valid, the next question we want to ask is: are the premises true? However, it is still important to focus on validity. The main reason is because each of us is limited in the amount of knowledge we have. For example, although John knows a great deal about art, he knows nothing about economics. As a consequence he can easily determine whether an argument about art is good in so far as he can determine that it is unsound because the premises are false. However, if he is assessing an argument in economics he will be unable to determine if the premises are true or false. As a consequence, the only thing that he can do to determine if the argument is good is attempt to determine if it is an instance of a valid argument form. That is he must attempt some sort of **content-independent** evaluation of the argument.

Chapter 5
The Language of Propositional Logic

5.0 Using formal systems to evaluate arguments

Validity is an important property of an argument. The reason why is that when an argument is valid we know that the pattern of reasoning it employs is *truth-preserving*. Validity of an argument tells us that the pattern of reasoning involved is such that it is impossible for the premises to be true and the conclusion false. As a consequence, the pattern of reasoning never takes us from actually true premises to an actually false conclusion; when an argument is valid, the truth of the premises guarantees the truth of the conclusion.

Although many of the arguments we encounter are given in natural languages, such as English, it turns out that when testing to see whether an argument is valid it is useful to employ a *formal system*. A formal language or system is an artificial language that has many properties in common with natural languages. A formal system is distinguished from a natural language by the fact that it is a language that humans do not acquire as a first language, and that is generally created in order to serve some specific purpose. Computers, for example, employ formal languages as part of the means by which they process information.

In the sections to come, a formal logical system, known as propositional logic (PL, for short), is going to be described. The system is going to be used to test whether certain kinds of arguments are valid. The arguments that will be considered are propositional arguments. In chapter 6, as noted before, we will introduce formal methods for testing the validity of a specific kind of quantificational argument known as *Categorical Syllogisms*.

The means by which we will test to see whether a natural language argument is valid within the formal system PL will involve three steps.

Step 1: Identify the argument in its natural language context.

Step 2: Translate the argument from natural language into the language of PL.

Step 3: Use some formal test procedure within the language of PL to determine if the argument is valid.

Notice that our procedure does not discuss soundness. The reason is that soundness has two components: the **formal** component and the **content** component. Formal logic, for the most part, can only help us determine whether an argument is valid. It cannot tell us, for a great many statements, whether the statement is true. For example, while logic can tell us whether an argument of the form.

If P, then Q. P. Therefore, Q.

is valid, it cannot tell us whether

'If a dog is animal, then a cat is an animal' is true.

In order to determine whether the claim above is true, we need to appeal to science, in particular, Biology.

In the procedure above, it is clear that the judgment that a specific argument in a natural language is valid depends heavily on Step 2. If the translation is bad, then the corresponding judgment is bad. Any time one translates into a formal system of any kind to test for validity, if the translation is poor or the formal system is itself inadequate to handle the kinds of statements involved in the argument, the judgment of validity based on it will be correspondingly weak. As a consequence, one has to pay careful attention when translating from a natural language into a formal language, just as one has to be careful translating from one natural language to another natural language when attempting to preserve meaning. And one has to be critical in their judgment of validity and invalidity based on the formal system they use by being aware of the limitations of logical analysis.

In order to understand propositional logic as a formal system and its relation to natural languages it is useful to first look at two natural languages, and the common properties between them. Here are some important general features of natural languages:

Symbols: the basic symbols of the system.

Syntax: the system of rules for generating complex or compound phrases from atomic units.

Semantics: the meaning of the atomic units, and the rules governing how the meanings of atomic units are put together to form complex meanings.

Concerning symbols, English and German have almost all of the same basic symbols captured by the overlap in their alphabet. However, English and Classical Arabic have completely different symbols captured by the disjointness of their basic alphabets. No part of English script is used in Classical Arabic.

Concerning semantics, it is clear that often two different languages will use different words to refer to the same thing. The English word 'snow' refers to snow; and the German word 'schnee' refers to snow. Across natural languages the atomic units, words, vary and which atomic units mean which things in the world vary.

Concerning syntax, amongst the various natural languages, some of the languages are subject-verb-object languages (SVO), and others are verb-subject-object languages (VSO). A subject first language is one in which the subject of the sentence is in first position given the reading/writing order. A verb first language is one in which the verb of the sentence is in first position.

English is an SVO language.

Consider the declarative sentence 'John is tall'. 'John' is the subject of the sentence. English is read and written from left to right, and so 'John' is also in first position. 'is tall' is the verb and is in second position.

In English it is against the syntactical and grammatical rules to write 'is tall John'. Given that 'is tall' is a verb phrase, and 'John' is a noun phrase, and the sentence is declarative. The rules of English syntax deem this combination of symbols ungrammatical.

Classical Arabic is a VSO language.

In classical Arabic the reading/writing order is from right to left, and one may state the verb of the sentence first, and then follow with the subject of the sentence.

جون طويل.

In the Arabic sentence above the punctuation mark is in the position where and English sentence would begin, the subject of the sentence is in second position, and the verb is in first position. And we are reading from right to left.

The main point that needs to be abstracted is that although the syntax is different between these two languages, both languages do have rules for determining how atomic units may be conjoined to make compound phrases. In English putting the verb first in a declarative sentence is incoherent. However, putting the verb first in classical Arabic is permitted.

Formal languages and systems are just like natural languages. Formal systems have a basic set of symbols. They have syntactical rules for determining which compound units are grammatical. And they have a semantic system, a set of rules for determining how the meaning of a compound unit is determined by the meaning of its atomic parts and mode of combination.

5.1 The syntax of propositional logic

Propositional logic is concerned with the logic of propositional connectives and operators. The main propositional operators and connectives that are studied in propositional logic are:

- Negation; *not*
- Conjunction; *and*
- Disjunction; *or*
- Material Conditional; *if….., then……*
- Biconditional; *…..if and only if……..*

These natural language phrases are used to change one proposition into another proposition, and to connect propositions in different ways.

Focusing on these operators and connectives forms the syntax of PL. These operators and connectives are the logical constants of propositional logic. Recall that abstracting away the propositional content, and holding constant the propositional operators and connectives generates propositional logic.

The symbols of PL are the following:

Capital letters: P, Q, R, S, T, U, V, W, X, Y, Z

> The English capital letters P through Z are statement letters of PL. They name specific statements depending on a scheme of abbreviation in which they are linked to a specific statement.

The unary propositional operator: '¬'
The binary propositional connectives: '→', '≡', '∧', '∨'
The grouping symbols: '(', ')', '[','\]'

In natural languages it is important to have a rule or procedure for determining which strings of symbols are grammatical and which strings are not. As we saw in the difference between Arabic and English, the syntactical rules of the language determine whether a sentence is grammatical or not. In the formal system PL, an atomic sentence or compound sentence is grammatical when it is a well-formed formula (wff, for short). In order to introduce this concept we first need to distinguish between *statement letters* and *statement variables*. The capital letters P……Z are part of the language PL, Just as the capital letters A….Z are part of the English Language.

For example, in translating the atomic English sentence

John is tall.

into PL, one may choose to represent that specific sentence by the use of the capital letter P. As a consequence, in PL, P means 'John is tall' relative to that *scheme of abbreviation*.

A scheme of abbreviation is an assignment of a statement of a natural language, such as English, to a statement letter of PL. A scheme of abbreviation is written by placing the capital letter to be used as an abbreviation of a sentences followed by ':', and the sentence itself. For example,

P: John is tall

tells us that the statement letter 'P' of PL stands for the English statement 'John is tall.'

However, sometimes it is necessary to talk about formulas of PL in general. It is necessary to talk about formulas in general because sometimes we need to talk about no particular formula, but a form that specific formulas share in common. In order to facilitate talking about forms of formulas lower case italicized letters are introduced.

Lower case letters: *p, q, r, s, t, u, v, w, x, y, z*

These letters are statement variables. They do not stand for any specific statement; rather they are variables ranging over every well-formed formula of PL. Unlike statement letters, statement variables are not part of the language of PL. They are part of the *meta-language* of PL. They are part of a language that is used for talking about the language of PL.

With the use of statement variables we can state precisely what a well-formed formula of PL is through a recursive definition. A recursive definition is one in which one part of the definition can be used along with another part of the definition in order to define a further element.

(1) Every statement letter P…. Z is a well-formed formula.

(2) If p and q are well-formed formulas, then

(i) $\neg p$,

(ii) $(p \wedge q)$,

(iii) $(p \vee q)$,

(iv) $(p \rightarrow q)$,

(v) $(p \equiv q)$,

are also well-formed formulas.

(3) Nothing is a well-formed formula unless rules (1) and (2) imply that it is.

The definition above tells us, for example that all of the following are not wffs of PL: (P¬Q), (∧QP), (→R). These strings are not wffs because they fail at least one clause of the definition above. By contrast ((P ∧ Q) ≡ R) is a wff because 'P', 'Q', and 'R' are all statement letters and so by (1) they are wffs; '(P ∧ Q)' is a wff by (1) and (ii), and '((P ∧ Q) ≡ R)' is a wff by (1), (ii), and (v).

One important consequence of the definition above is that we can arrive at a procedure for determining for any given wff what kind of formula it is. In PL every compound formula is either a

- Negation
- Conjunction
- Disjunction
- Conditional
- Biconditional

These options correspond to (i) - (v). To determine what kind of formula a given formula is one need only isolate the *main connective or operator*. Since atomic formulas can by joined by the operators and connectives of PL to form compound formulas, and this process can be repeated over and over again, it is important to identify a formula by the main connective, that is the connective that has every other formula as a component of it.

Examples:

$(P \rightarrow (Q \wedge R))$

The main connective of the formula is $\rightarrow$ because the conjunction $(Q \wedge R)$ is only one part of the conditional.

$((P \rightarrow R) \wedge (R \rightarrow P))$

The main connective is $\wedge$ because the two formulas with $\rightarrow$ in them occur as parts of the conjunction.
$\neg(P \rightarrow (R \wedge Q))$

The main operator is $\neg$ because both the formula involving $\rightarrow$ and $\wedge$ occur as parts of the negation.

A Convention Concerning the Grouping of Formulas

It is important to note that given the definition of a well-formed formula the following formulas are strictly speaking not wffs.

(a) $P \rightarrow Q$
(b) $P \vee (Q \wedge R)$
(c) $(P \equiv Q) \wedge R$

(a) – (c) are not wffs because they lack the correct number of parenthesis. In order to satisfy the letter of the definition above they would have to be rendered as follows:

(d) $(P \rightarrow Q)$

(e) $(P \vee (Q \wedge R))$
(f) $((P \equiv Q) \wedge R)$

In general, each connective and operator requires its own set of parentheses. Every time a binary connective is introduced or a negation is used one should employ a set of parentheses demarcating the statement letters or compound formulas it operates on.

Although this is a strict rule of the syntax of PL, we will be introducing a convention where by one can drop the last set of parentheses.

So, though, (a) – (c) are technically deficient, they are in fact coherent. There is no ambiguity in what is said. In section 2.6 and 2.7 when we discuss translation there will be more conventions concerning grouping, however, those rules depend on the specific operator being used. As of now, the only convention introduced is that outer most parentheses may be dropped. One should be careful in how they understand this. For example:

(g) $P \rightarrow Q$
(h) $\neg(P \wedge Q)$
(i) $\neg P \wedge Q$

While (g) is coherent and the same as $(P \rightarrow Q)$, (h) and (i) are not the same. (h) is a negation, and (i) is a conjunction. In (h) the negation governs the whole formula, in (i) conjunction governs the whole formula. Our convention allows us to drop the outer most parentheses when the main connective is a binary operator. When the main connective is a unary operator, such as negation, one cannot drop the outer parentheses unless the parentheses encompass the negation operator such as:

(j) $(\neg P \wedge Q)$

can be represented by our new convention as

(k) $\neg P \wedge Q$

Finally, to touch base with what was said earlier, just as different natural languages have different grammars, such as that generated by (SVO) vs. (VSO) difference, it is important to note that logical systems have been developed, and can be developed, which have different rules for constructing well-formed formulas. Perhaps, one of the most interesting cases is Polish notation. In Polish notation, one places the binary connective before the statement letters. The compound statement form ($p \land q$), which is a well-formed formula of PL, is the same compound statement form as K (pq) in Polish notation. In Polish notation, 'K' is the same propositional connective as '∧' of PL. Polish notation differentiates itself from PL by the fact that connectives and operators come in right to left reading order sequentially before the statements they join. Operators are out in front of the statements they operate on, just as in (VSO) languages the verb is in front of the subject and object it operates on.

5.2 The Semantics of Propositional Logic

'¬', '∧', '∨', '→', '≡' are symbols of PL and not of English. According to the syntactic rules of PL, well-formed formulas can be constructed by combining the symbols above with statement letters. Well-formed formulas of PL include:

$((P \rightarrow Q) \land (R \lor S))$

$(\neg R \lor (Q \rightarrow S))$

In addition to the syntactic rules that govern how statement letters, and operators and connectives can be combined to form well-formed formulas of PL, PL also has semantic rules. The semantic rules of a language govern how the meaning of its constituent parts and their mode of combination determine the meaning of a compound statement.

The connectives and operators of PL are called **truth-functional operators**. They are called truth-functional operators because their semantical role is given by the way they map truth-values to truth-values. Each operator of PL determines what the truth-value of a compound statement involving the operator is depending on what the truth-value is of the formulas making up the compound. A truth functional operator is just like an algebraic function, such as x^2. The function x^2 takes numbers as inputs and yields numbers as output. Put in 2 and you get 4, put in 3 and you get 9. The function x^2 is different from the function x^3 because the range of each function is different. When you put in 2 to x^2 you get 4, but when you put 2 into x^3 you get 8. Truth-functions in propositional logic are similar. The only difference is that the inputs are either truth or falsity, and the output is either truth or falsity.

In the system PL' there are only two truth-values: *Truth* and *Falsity*. The letter 'T' is used to represent Truth, and the letter 'F' is used to represent Falsity. The definition of each operator is given by a truth-table definition. A truth-table is a table that shows what truth-value a compound statement has given an initial set of truth-values. The truth-table lists all the possible combinations of truth-values and what the value of the compound statement is for each possibility. We will discuss truth-table analysis in more detail later.

For now the important thing to note is that in the columns where the operator is not present, there is a list of all the possible truth-values that can be distributed over the component parts. And in the column where the operator is present, there is a truth-value for the operator given the input values in the row.

The truth-table definition for '¬' is the following:

p	**¬ *p***
T	**F**
F	**T**

The definition tells us for any arbitrary formula *p* what the value of ¬*p* is. Since negation is a unary operator there are only two possibilities represented under the column for *p*. The definition simply tells us that ¬*p* has the opposite value of *p*. This definition tells us that the symbol of PL '¬' is such that if it is adjoined to a formula that is true, it makes the formula false; and if it is attached to a formula that is false, it makes the formula true.

The truth-table definition for '∧' is the following:

p	q	$p \wedge q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

The definition tells us for any arbitrary formulas p and q what the value of $(p \wedge q)$ is. Since conjunction is a binary connective there are four possibilities represented under the columns for p and q. The definition tells us that $(p \wedge q)$ is true only when both conjuncts are true. The '∧' symbol of PL has the function of either taking two formulas that are true and making a compound formula that is true, or taking two formulas where at least one is false and making a compound formula that is false.

The truth-table definition for '∨' is the following:

p	q	$p \vee q$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

The definition tells us for any arbitrary formulas p and q what the value of $(p \vee q)$ is. Since disjunction is a binary connective there are four possibilities represented under the columns for p and q. The definition tells us that $(p \vee q)$ is false only when both disjuncts are false. The '∨' symbol of PL has the function of either taking two formulas that are false and making a compound formula that is false, or taking two formulas where at least one is true, and making a compound formula that is true.

The truth-table definition for '→' is the following:

p	q	$p \rightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

The definition tells us for any arbitrary formulas p and q what the value of $(p \rightarrow q)$ is. Since the material conditional is a binary connective there are four possibilities represented under the columns for p and q. The definition tells us that $(p \rightarrow q)$ is false just in case the p is true, and q is false. The '$\rightarrow$' symbol of PL has the function of making a true compound formula out of any two formulas, as long as it is not the case that the first formula is true and the second formula false.

The truth-table definition for '$\equiv$' is the following:

p	q	$p \equiv q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

The definition tells us for any arbitrary formulas p and q what the value of $(p \equiv q)$ is. Since the biconditional is a binary connective there are four possibilities represented under the columns for p and q. The definition tells us that $(p \equiv q)$ is true just in case both p and q are true or both p and q are false. The '$\equiv$' symbol of PL takes any two formulas that have the same truth-value, and makes a compound formula that is true. If the two formulas joined by the triple bar have opposite truth-values, then the compound formula is false.

It is important to recognize and keep in mind that these symbols are formal symbols of the language of PL. These symbols have the definitions that are given above. Whether or not any term of a natural language, such as English, operates in the way that the symbols above do is an interesting and pressing philosophical question. We will get more into this in the sections on translation that follow.

What the semantics of PL tells us is that the operators and connectives of the language function in a combinatorial role. Given that some formula or set of formulas has a certain value, operators and connectives of the language generate a further value. Going in the other direction, given any formula with a connective or operator involved, there will be a systematic way of decomposing and analyzing it.

Chapter 6
Translation into Propositional Logic

6.1 *Negations, Conjunctions, and Disjunctions*

6.1.1 Negation

If I assert 'it is not the case that France won the 2006 World Cup,' what I assert is that the statement 'France won the 2006 World Cup' is false.

The English phrases:

It is not the case that……
It is not true that……
It is false that ……

and embedded negations, such as

….is not ……
….does not….
…… no……..

all involve the negation operator. The negation operator is a unary operator. It operates on a single statement (atomic or compound) to form a new statement. The point of the negation operator is to assert the opposite truth value of the embedded statement. English sentences involving negation get translated into PL by the '¬' symbol. Here are some examples:

It is not the case that John is tall.	*John is not running.*
P: John is tall.	P: John is running.
$\neg P$	$\neg P$

Mary does not own a house.	*It is not true that Mary is short*.
Q: Mary owns a house.	Q: Mary is short.
¬Q	¬Q

In all of the examples above, the scheme of abbreviation takes the affirmative statement, rather than the whole statement so that the logical structure of the original statement is revealed. In translating, we always want to reveal as much logical structure as possible given the constraints of the formal system we are working in. Since PL has a symbol for negation, '¬', we ought to take out a negation if it is present in a statement, and correctly captured by the meaning of '¬'.

For example, it would be improper to translate

It is not true that Italy is a better soccer team than Germany.

as

P: It is not true that Italy is a better soccer team than Germany.

This translation does not adhere to the maxim that we can reveal the logical structure of the negation present in the statement.

It is important to note that not every occurrence of 'not', 'non', 'no' in an English sentence is a negation of the kind captured by '¬', and that various prefixes, such as 'un-', 'ir-', 'in'', 'im', and 'a-' can signal negation. The mark of a true negation is that there is no third or middle option.

For example,

It is impossible that 2 + 2 = 5.

means

It is not the case that it is possible that 2 + 2 = 5.

However,

It is immoral to torture babies.

does not mean

It is not the case that it is moral to torture babies.

rather

It is immoral to torture babies.

means

It is wrong to torture babies.

Since some actions are neither moral nor immoral, but simply amoral --having no morality pertaining to them--, such as tripping on the sidewalk, one has to be careful when translating "immoral".

In order to test whether a statement that potentially involves negation actually involves negation simply ask the following question:

Is there a middle option available for the term in question? If there is, then the occurrence of the prefix potentially expressing negation, may not express true binary negation. If there is not, then the occurrence of prefix potentially expressing negation, probably is expressing true binary negation.

More importantly, when one is translating a statement into PL, which potentially involves negation, one has to pay attention to the context in which it occurs for clues as to what is really being said.

6.1.2 Conjunction

The English word

And

conjoins two statements to form a new statement. When one asserts a compound sentence joined by ‘and’ one affirms both conjuncts.

Asserting: *Mary went to school and John went to school.*

is affirming both

Mary went to school.

John went to school.

When one makes a conjunction, their intention is to assert both conjuncts. As a consequence, a conjunction is true only when both conjuncts are true, and false just in case at least one conjunct is false.

In English, if one asserts

Mary went to school and John stayed home.

If it is true that Mary went to school, but false that John stayed home, then it is false that

Mary went to school and John stayed home.

Conjunctions are translated into PL with the symbol ‘∧’ between the two formulas either atomic or compound.

A conjunction joins atomic or compound statements together to form a new statement. The new statement affirms both conjuncts.

Translations

John went to school and Mary went to school.

P: John went to school. Q: Mary went to school.

$(P \wedge Q)$

John is tall, Mary is tall, and Bill is tall.

P: John is tall. Q: Mary is tall. R: Bill is tall.

$(P \wedge Q) \wedge R)$

Both John and Mary passed the test.

P: John passed the test. Q: Mary passed the test.

$(P \wedge Q)$

John and Mary both failed the test.

P: John failed the test. Q: Mary failed the test.

$(P \wedge Q)$

However, 'and', 'both…..and….', and their variants are not the only devices of the English language that are conjunctions. In addition, some instances of 'and', 'both…..and….', and their variants are not simply conjunctions. Here are some common phrases that often express conjunction.

But	Yet	Moreover	Whereas
Although	Nevertheless	However	

These expressions can be used to join two statements together by affirming both statements, the difference between each of them, and ‘and’ is the shading of the statements affirmed.

Taxes are going up, but housing prices are falling.

affirms that both

Taxes are going up.

Housing prices are falling.

just as

Taxes are going up, and housing prices are falling.

The difference is that ‘but’ is a discounting phrase, while in the context above ‘and’ is neutral in shading, it does not imply a difference in how a preference for either statement. The word ‘but’, by contrast, signals that the author affirms the first conjunct of the compound statement, and discounts it in favor of the truth of the second conjunct.

Consider the sentence in a larger context.

Look! It is true that taxes are going up, but housing prices are falling. So, things are not that bad.

The author grants that taxes are going up, and asserts that things are not as bad as they would be if housing prices were going up or staying the same.

Just as in the case of negation one must pay attention to context in translating conjunctions. There are various instances in which more or less is being affirmed by the use of ‘and’ than the affirmation of two independent statements. In cases like these one should note that translating the statement into PL by the use of ‘$\wedge$’ will create a loss of meaning.

Here are some common cases:

Temporal flow –

Sometimes uses of ‘and’ between two sentences or between two parts of speech signals a temporal flow of events in addition to affirmation of both conjuncts.

Axel Rose’s voice went out, and the crowd threw food at the stage.

does not just affirm that

Axel Rose’s voice went out.

The crowd threw food at the stage.

It also affirms the statements in a temporal sequence where Axel Rose’s voice went out, *and then* the crowd through food at him.

If one thinks that a particular instance of ‘and’ involves temporal flow one can test for it by switching the order of the sentence.

For example

The crowd threw food at the stage, and Axel Rose’s voice went out.

Although this could have been true, it does not convey the same information that the former sentence does.

‘and’ also can cause problems when it is used between parts of speech.

Johnny discovered the cure for cancer and became famous.

Does not just affirm that ‘Johnny discovered the cure for cancer’ and Johnny became famous’. If the sentence were rewritten as ‘Johnny became famous and discovered the cure for cancer’ it would fail to signal that the reason Johnny became famous is because he discovered the cure for cancer.

Collective Subject –

Some uses of ‘and’ signal that a collective subject exists, and that an action is being performed collectively.

Jane and Bill danced the night away.

Does not just mean

Jane danced the night away, and Bill danced the night away.

It is true that the sentence does mean that both Jane and Bill danced the night away. However, it also conveys the further information that

Jane and Bill danced the night away <u>together</u>.

Collective subject also occurs in the following:

June, July, and August make up the summer recess.

Bill and Tom pushed the car.

France, England, and America conquered Japan and Germany during WWII.

Additive comparison

Some times ‘and’ is used to conjoin two things that are then being compared to a single thing.

New York City is bigger than Boston and Philadelphia.

The use of ‘and’ could be a conjunction if what is meant is that

New York City is bigger than Boston.

and

New York City is bigger than Philadelphia.

However, one could also mean the following

New York City is bigger than Boston and Philadelphia <u>put together</u>.

In this case 'and' is used not to conjoin two statements, but to conjoin two things in an additive manner.

This can occur also in

Bill is shorter than Mary and Jane.

but generally not in

Bill is shorter than both Mary and Jane.

In the second case the additive is eliminated by the presence of 'both'.

'Both' signals that

Bill is shorter than Mary and Bill is shorter than Jane.

In most cases it is not hard to tell whether 'and' is being used conjunctively or non-conjunctively. The key is to look for signals present in the text or utterance, and apply tests that help determine whether a use of 'and' is conjunctive. Often it is useful to attempt to switch the direction or add in 'both' in order to help hear if the statement is conjunctive.

6.1.3 Disjunction

The English word

Or

connects two statements together as a disjunction.

If I assert that

Either Bill will run for the election or John will run for the election.

I assert that either the either the first disjunct is true

Bill will run for the election.

or the second disjunct is true

John will run for the election.

A disjunction is true just as long as one of the disjuncts is true, and false when both of the disjuncts are false.

Disjunctions are translated into PL by the use of '∨'.

Translations

Juventus players will go to other nations or Juventus players will stay in Italy.

P: Juventus players will go to other nations; Q: Juventus players will stay in Italy.

$(P \vee Q)$

Either Lillian Thuram will play for Chelsea or FC Barcelona.

P: Lillian Thuram will play for Chelsea; Q: Lillian Thuram will play for FC Barcelona.

(P ∨ Q)

Disjunctions also fall into two types: **inclusive** and **exclusive**.

In an inclusive disjunction one intends that both disjuncts can be true. In an exclusive disjunction one intends that only one disjunct can be true. In both cases however, the disjunction is false if both disjuncts are false.

Suppose, as did happen, Italy plays France in the 2006 World Cup, and someone says,

Either France will win the World Cup or Italy will win the World Cup.

The sense of disjunction that is being employed here is exclusive. In a no draw final match it is impossible for both teams to win, so at least one of the disjuncts is true, but also <u>at most one is true</u>.

Contrast the sentence above with the following.

Suppose France is playing Portugal in the semifinals, and Italy is playing Germany in the semifinals, and someone says,

Either France will go to the World Cup or Italy will go to the World Cup.

The sense of disjunction that is employed here is inclusive. It is possible that France beats Portugal, and Italy beats Germany, and both meet in the World Cup. What is intended is that at least one of the conjuncts is true, but it is not intended that at most one conjunct is true.

The following is another case of exclusive disjunction:

Dinner comes with either soup or salad.

One cannot, simply by ordering dinner, have both soup and salad. Determining whether an occurrence of ‘or’ or ‘either…or…’ is inclusive or exclusive requires investigating the context of the text or utterance, and applying certain tests.

One kind of test is the *either-or-but-not-both* test

Sometimes thinking about the two statements that make up the disjunction in by filling in the blanks in ‘either….or….but not both’ one can determine whether the original ‘or’ is exclusive. Usually, one has to employ additional background information for help. If one can change a pure ‘either…or…’ statement into an ‘either….or…but not both’ statement without adding meaning, then the statement is an exclusive or. If one cannot change it without adding an unintended meaning, then the statement is an inclusive *or*.

Although there is a difference between exclusive and inclusive disjunction, in PL ‘∨’ is defined to match inclusive disjunction. Exclusive disjunction can be defined as a special case.

Even though ‘or’ and ‘either….or’ are the main English phrases for disjunction there are other phrases that are best translated as a disjunction.

Perhaps the most interesting one in this category is ‘unless’.

John will go to school unless the bus breaks down.

P: John will go to school; Q: The bus breaks down.

$(P \vee Q)$

The word ‘unless’ functions in English as a disjunction in much the same way that ‘but’ functions as a conjunction. ‘but’ is just like ‘and’ except that it shades the two statements it conjoins. 'unless' is a disjunction, except for the fact that it shades the two statements it disjoins, the world 'unless' often signals exception. When 'unless' occurs between two statements it often signals that the first thing is going to happen and should happen, and only on the chance that something else happens the first event won't happen. 'Unless' can also occur at the beginning of a sequence of statements.

Unless housing prices go down, people will be homeless.

P: Housing prices go down; Q: People will be homeless.

$(P \vee Q)$

In this case 'unless' does not signal exception, it signals causation. Housing prices staying the same will cause people to be homeless.

Although there are multiple sense in which unless functions in English, many can be accurately translated into PL by the use of '$\vee$'.

6.1.4 Commutativity and associativity of conjunction and disjunction

Both 'and' and 'or' have the property of being commutative and associative when iterated without other binary operators, such as material conditional.

John went to school and Mary went to school.

means the same as

Mary went to school and John went to school.

So, using the scheme of abbreviation

P: John went to school; Q: Mary went to school.

$(P \wedge Q)$ means the same as $(Q \wedge P)$.

Likewise,

John went to school or Mary went to school.

gets translated under the same scheme of abbreviation as

$(P \vee Q)$, which means the same as $(Q \vee P)$.

The binary connectives '$\wedge$' and '$\vee$' are commutative, just as 'and' and 'or' are commutative.

A good example of commutation is addition.

2 + 3 = 5 is the same as 3 + 2 = 5.

Subtraction is not commutative.

3 - 2 = 1, but 2 – 3 = -1

'and' and 'or' are also associative. A good example of associativity is multiplication.

2 * (3 * 5) = 30; and (2 * 3) * 5 = 30.

When there is no other operator, such as addition or subtraction, present order of multiplication does not matter.

Likewise, when there is no other binary connective present other than disjunction or conjunction alone, order of disjunction and conjunction does not matter.

For example:

$(P \wedge Q) \wedge R)$

means the same as

$(P \wedge (Q \wedge R)$

and

$(P \vee Q) \vee R)$ means the same as $(P \vee (Q \vee R)$

6.1.5 Grouping: mixed operators and connectives.

Often disjunction, conjunction, and negation occur together in compound constructions such as:

It is not the case that both Italy loses to the US and Switzerland beats France.

Neither Portugal nor Spain will beat France.

It is not the case that both Germany and Argentina will advance to the semifinals or that Portugal will beat France.

Neither France nor Germany will advance to the semifinals unless Italy and Ghana beat the US in group play.

In cases like these, it is very important how statements are grouped using parentheses, and where negation is placed relative to conjunction and disjunction.

Translations

It is not the case that both Italy loses to the US and Switzerland beats France.

P: Italy loses to the US; Q: Switzerland beats France. The statement is a negation of a conjunction, $\neg(P \wedge Q)$. One should not confuse the negation of a conjunction with a conjunction of negations. $(\neg P \wedge \neg Q)$ is not a correct translation of the statement because it says, “it is not the case that Italy will lose to the US and it is not the case that Switzerland beats France," which is a conjunction of two negated statements. P and Q should be grouped together and the negation applied to the compound statement.

Neither Portugal nor Spain will beat France.

P: Portugal will beat France; S: Spain will beat France.
The statement is a negation of a disjunction, $\neg(P \vee S)$. One should not confuse the negation of a disjunction with a disjunction of negations. $(\neg P \vee \neg Q)$ is not the correct translation of the statement because it says, "either it is not the case that Portugal will beat France or it is not the case that Spain will beat France," which is a disjunction of two negated statements. P and Q should be grouped together with a disjunction, and the negation applied to the compound statement.

It is not the case that both Germany and Argentina will advance to the semifinal or that Portugal will beat France.

P: Germany will advance to the semifinal; Q: Argentina will advance to the semifinal; R: Portugal will beat France. The statement is a disjunction, which has one conjunction that is negated as a disjunct, $\neg(P \wedge Q) \vee R)$

Neither France nor Germany will advance to the semifinals unless Italy and Ghana beat the US in group play.

P: France will advance to the semifinals; Q: Germany will advance to the semifinals; R: Italy beats the US in group play; S: Ghana beats the US in group play. The statement is a disjunction that has a negated disjunction as one disjunct, and a conjunction for the other disjunct. $\neg(P \vee Q) \vee (R \wedge S)$

In general, translating compound statements involving negation and disjunction, negation and conjunction, or negation and both conjunction and disjunction requires paying attention to where the negation is being applied. Here are some general constructions, and their common translations:

English Phrase	Logical Translation
Not both …… and ……	$\neg(p \wedge q)$
Neither ……nor……	$\neg(p \vee q)$
Both not …… and ……	$(\neg p \wedge \neg q)$

6.2 *Material Conditionals and Biconditionals*

6.2.1 Material Conditionals

Material Conditionals are most commonly expressed in English by the use of:

If……, then…….

However, there are three central variations on this construction

……..,if……

Only if………,……..

………only if……..

Material conditionals are constituted out of two parts: the *antecedent* and the *consequent*. The antecedent of a conditional is the *if*…… part of the construction. The consequent of a conditional is the *then….* part of the conditional.

If …………., then…………..

Antecedent Consequent

The antecedent of a material conditional expresses a sufficient condition. The consequent of a conditional expresses a necessary condition. The distinction between the two can be shown via examples:

(a) Getting an A on the final exam is a necessary condition for getting an A in the class.

(b) Getting a B on all of the exams is a sufficient condition for getting a B in the class.

(a) is an example of a necessary condition because *unless* one gets an A on the exam, it is impossible to get an A in the class.

(b) is an example of a sufficient condition because *even though* getting a B on all of the exams will get you a B in the class, it is not the *only* way to get a B in the class. One could get an A, B, and C, which would average to a B.

One way to remember that the antecedent of a material conditional expresses a sufficient condition and the consequent expresses a necessary condition is by the acronym SUN.

Sufficent ⊃ Necessary, where the '⊃' stands for 'if.....then.....'
Material conditionals are translated into PL by the use of the '→'. A material conditional is false only when the antecedent is true and the consequent false.

Translations

If Zidane passes to Ribery, then Sagnol will close the center.

P: Zidane passes to Ribery; Q: Sagnol will close the center.

(P → Q)

If taxes go up, then inflation will rise.

T: Taxes go up; R: Inflation will rise.

(T → R)

Iran will supply arms to Syria only if Syria helps Hezbollah.

R: Iran will supply arms to Syria; S: Syria helps Hezbollah.

(R → S)

Only if Jenna passes the exam, John will join the army.

P: Jenna passes the exam; Q: John will join the army.

(Q → P)
Zidane will pass to Sagnol, if Ribery goes up the middle.

Z: Zidane will pass to Sagnol; R: Ribery goes up the middle.

(R → Z)

Here are the central constructions and their translation in PL.

Construction	Translation
If p, then q	$(p \rightarrow q)$
p, if q	$(q \rightarrow p)$
p only if q	$(p \rightarrow q)$
Only if p, q	$(q \rightarrow p)$

6.2.2 Biconditionals

Biconditionals are generally expressed in English by the phrase

……if and only if…..

However, there are variations of it as well, such as

…….just in case….

A biconditional is a conjunction of two material conditionals. For example,

John will join the army if and only if John passes the exam.

is actually the conjunction of the following material conditionals.

If John joins the army, then John passes the exam.

If John passes the exam, then John will join the army.

Biconditionals are translated into PL by the symbol '≡', and are true just in case each part is true or each part is false, and never when only one part is true.

Translations

Zidane will shoot if and only if Ribery passes to him.

Z: Zidane will shoot; R: Ribery passes to Zidane.

$(Z \equiv R)$

Frank will run in the next election just in case Mary will run in the next election.

P: Frank will run in the next election; Q: Mary will run in the next election.

$(P \equiv Q)$

Mary will win the award if and only if Lisa votes.

R: Mary will win the award; S: Lisa votes.

$(R \equiv S)$

6.2.3 Grouping: commutativity and associativity

The biconditional like conjunction and disjunction is commutative and associative. $(P \equiv Q)$ is the same as $(Q \equiv P)$; and $(P \equiv (Q \equiv R)$ is the same as $(P \equiv Q) \equiv R)$. The reason a biconditional is commutative and associative is because a biconditional just is a *conjunction* of two conditionals. So, if conjunctions are commutative and associative, so are biconditionals.

However, material conditionals are neither commutative nor associative. $(P \rightarrow Q)$ is not the same as $(Q \rightarrow P)$; and $(P \rightarrow (Q \rightarrow R)$ is not the same as $(P \rightarrow Q) \rightarrow R)$.

Consider the following:

Z: Zidane will charge forward; R: Ribery will play defense; S: Sangol will run the midfield.

$(Z \rightarrow R)$ says that 'if Zidane charges forward, Ribery will play defense'; which does not mean the same thing as $(R \rightarrow Z)$, which says that If Ribery plays defense, Zidane will charge forward.

In the $(Z \rightarrow R)$ situation, Ribery plays defense because Zidane charges forward. In the $(R \rightarrow Z)$, case Zidane charges forward because Ribery plays defense.

Likewise, with the difference between

$(Z \rightarrow R) \rightarrow S)$ and $(Z \rightarrow (R \rightarrow S)$

$(Z \rightarrow R) \rightarrow S)$ says, that 'If Zidane will charge forward, only if Ribery will play defense, then Sangol will run the midfield'. $(Z \rightarrow (R \rightarrow S)$ says that 'If Zidane charges forward, then Ribery will play defense only if Sagnol runs the midfield. The two statements are very different.

Translations for grouping with mixed operators

It is not the case that if the Rangers win the Stanley cup, the Oilers will trade Gretzsky.

P: The Rangers win the Stanley cup; Q: The Oilers trade Gretzsky.

$\neg(P \rightarrow Q)$

If France does not win the World Cup, then Italy will win the World Cup.

P: France wins the World Cup; Q: Italy wins the World Cup.

$(\neg P \rightarrow Q)$

Either France or Togo will advance from group play if Switzerland gets 4 points.

P: France advances from group play; Q: Togo advances from group play; R: Switzerland gets 4 points.

$(R \rightarrow (P \vee Q)$

Germany and Argentina will advance from group play only if Poland and Spain get 4 points.

P: Germany will advance from group play; Q: Argentina will advance from group play; R: Poland gets 4 points; S: Spain gets 4 points.

$(R \wedge S) \rightarrow (P \wedge Q)$

Ghana beats the US if Dempsey does not play.

P: Ghana beats the US; Q: Dempsey plays.

$(\neg Q \rightarrow P)$

Foreign relations with the Middle East will improve only if gas prices do not increase.

P: Foreign relations with the Middle East will improve; Q: Gas prices increase.

$(P \rightarrow \neg Q)$

Only if foreign relations with the Middle East improve will gas prices not increase if there is a cease fire.

P: Foreign relations with the Middle East improve; Q: Gas prices increase; R: There is a cease fire.

$(R \rightarrow \neg Q) \rightarrow P)$

Unit 3
Formal Techniques of Analysis I

Chapter 7
Truth-Table Analysis

7.1 Truth-Tables and the Logical Status of Propositions

Often it is important to determine what the logical status of a proposition is. For example,

Either the wall is red or it is not red.

The wall is red and it is not red.

The wall is red.

Each has a distinct logical status. The logical status of a proposition is determined by what kind of truth-table it has.

Logical Status	Meaning	Truth-Table
Contradiction	Always False	Every row F
Contingent	Dependent	Some row T & Some row F
Tautology	Always True	Every row T

In addition, it is also useful to determine what the logical relation is between propositions. For example,

Both England and France will not advance out of group play.

and

Neither England nor France will advance out of group play.

mean the same; just as

If England wins, France will win.

means the same as
If France does not win, then England will not win.

But

If France wins, then England will not win.

does not mean

France will win and England will win.

Here are three relations a group of propositions can bear to one another:

Logical Status	Meaning	Truth-Table
Equivalent	Same Truth-Value Distribution	The truth-tables are identical
Satisfiable/Consistent	Possibly all True	There is some row where all propositions are true
Unsatisfiable/Inconsistent	Not Possibly all True	No row where all propositions are true

By using truth-tables, it is simple to determine what the logical status of a proposition or a group of propositions is.

To use a truth-table to determine what the logical status of a single proposition is one must construct a truth-table-computation.

7.2.1 Truth-tables and general rules of construction

A truth-table consists of rows and columns. The rows go from left-to-right, and the columns go top-to-bottom. A truth-table needs at least as many columns as

- Statement letters
- Non-atomic formula
- The formula to be computed

and exactly 2^N rows; where N is the number of atomic statement letters in the formula. To construct a truth-table for $((P \wedge Q) \rightarrow \neg R)$ one would need 8 rows (since there are 3 atomic statement letters, so N=3 and $2^3 = 8$), and 5 columns. The 8 rows exhaust all the possible combinations of truth-value assignments to all of the atomic statement letters. The 5 columns are a consequence of having one column for every statement letter, and one column for each non-atomic formula, which is not the formula being computed. The final column is the column of the formula being computed. We will refer to that column as column 6 in this case

	1	2	3	4	5	(6) Formula
0	P	Q	R	$\neg R$	$(P \wedge Q)$	$(P \wedge Q) \rightarrow \neg R)$
1	T	T	T			
2	T	T	F			
3	T	F	T			
4	T	F	F			
5	F	T	T			
6	F	T	F			
7	F	F	T			
8	F	F	F			

Columns (1), (2), and (3) are for the atomic statement letters. Columns (4) and (5) are for the non-atomic formulas that are sub-components of the main formula being computed. The final column is the formula whose value is being computed.

In addition, there is also a general procedure for placing every possible combination of truth-values for the number of atomic statements. Where there are three atomic statement letters the procedure beings by starting with column (1) fill the first 4 rows with T, and the next 4 rows with F. With column (2) fill the first two rows with T, the next two rows with F, and continue alternating every two rows until the end. With column 3, start with T in the first row, and alternate T and F until the end. The general principle being followed is that alternation between successive Ts and Fs halves as one goes from left-to-right along the columns.

In order to compute the values for the rows in column (4) one must apply the truth-table for ¬ to column (3). Since the truth-table for ¬ instructs us to simply give the opposite truth-value to the initial truth-value, one simply puts the opposite truth-value in column 4 for each row given column 3. Column (3) is the set of initial values that the ¬ table is applied to. In order to compute the values for the rows of column (5) one must apply the truth-table for ∧ to the values in columns (1) and (2). The table for ∧ tells us that a conjunction is true only when both conjuncts are true. So, the only rows in column 5 that get a T in them are ones where there is a T in both column (1) and (2). In order to compute the value of any given row in the final column one simply looks at the truth-table for → which tells us that any time the antecedent is true and the consequent false the formula containing → is false. In the case of (P ∧ Q) → ¬R), (P ∧ Q) is the antecedent, and ¬R is the consequent. So, any time a cell in column (4) has an F, and the corresponding cell in column (5) has a T, column (6) gets an F. In all other cases, the column (6) gets a T.

	1	2	3	4	5	(6) Formula
0	P	Q	R	¬R	(P ∧ Q)	(P ∧ Q) → ¬R)
1	T	T	T	F	T	F
2	T	T	F	T	T	T
3	T	F	T	F	F	T
4	T	F	F	T	F	T
5	F	T	T	F	F	T
6	F	T	F	T	F	T
7	F	F	T	F	F	T
8	F	F	F	T	F	T

7.2.2 Determining the Logical Status of a Single Proposition

As noted before sometimes it is useful to determine the logical status of a proposition. There are three options. A proposition can either be contradictory, tautologus, or contingent.

Example: $(P \wedge \neg P)$

	1	2	3
0	P	$\neg$P	$(P \wedge \neg P)$
1	T	F	F
2	F	T	F

Column (3) is computed by applying the $\wedge$ truth-table to columns (1) and (2). Since the $\wedge$ truth-table tells us that a conjunction is true only when both conjuncts are true, and in the case above P is the only statement letter in the conjunction every row of the table will be false.

The logical status of $(P \wedge \neg P)$ is *contradictory*. The statement is never true.

Example: $(\neg P \rightarrow P)$

	1	2	3
0	P	$\neg$P	$(\neg P \rightarrow P)$
1	T	F	T
2	F	T	F

Since $(\neg P \rightarrow P)$ only has one statement letter the truth-table only requires one column for P and two rows for the two possible truth-values of P. Column (2) is the $\neg$ truth-table applied to column (1). Column (3) is the $\rightarrow$ truth-table applied to column (1) and (2). The logical status of $(\neg P \rightarrow P)$ is *contingent*, since row (1) has a T, and row (2) has an F.

Example: $(P \vee \neg P)$

	1	2	3
0	P	$\neg$P	$(P \vee \neg P)$
1	T	F	T
2	F	T	T

Column (3) is computed by applying the ∨ truth-table to columns (1) and (2). Since the ∨ truth-table tells us that a disjunction is false only when both disjuncts are false, and in the case above P is the only statement letter in the disjunction every row of the table will be true.
The logical status of (P ∨ ¬P) is *tautologus*. The statement is always true.

Example: (P ∧ ¬Q)

	1	2	3	4
0	P	Q	¬Q	(P ∧ ¬Q)
1	T	T	F	F
2	T	F	T	T
3	F	T	F	F
4	F	F	T	F

Column (1) is the total set of truth-values for P. Column (2) is the total set of truth-values for Q. Column (3) is the computed value of ¬Q, determined by the truth-table for ¬ applied to the column (2). Column (4) is the computed value of (P ∧ ¬Q), determined by the ∧ table applied to column (1) and (3). The logical status of (P ∧ ¬Q) is *contingent* because at least one row in column (4) has a T, and at least one has an F.

7.2.3 Determining the logical status of a group of propositions

When determining the logical status of a group of propositions the procedure is roughly the same as it is in the case of determining the logical status of a single proposition. Of course the truth-table gets more complex.

Example: (P ∧ Q), ¬ (¬P ∨ ¬Q)

	1	2	3	4	5	6	7
0	P	Q	¬P	¬Q	(P ∧ Q)	(¬P ∨ ¬Q)	¬ (¬P ∨¬Q)
1	T	T	F	F	T	F	T
2	T	F	F	T	F	T	F
3	F	T	T	F	F	T	F
4	F	F	T	T	F	T	F

We want to compare two formulas: (P ∧ Q) and ¬(¬P ∨ ¬Q) to see what the logical relation is between them. First, we put down all the possible combinations of truth-values for our atomic statement letters P and Q in columns (1) and (2). Second, we compute the value of columns (3) and (4) by applying the negation operator to column (1) for column (3), and column (2) for column (4). Third we compute the values for (5) and (6) based on the values in (1), (2), (3), and (4) and the appropriate truth-table for the operator we are computing. Column (5) uses conjunction. Column (6) uses disjunction. Finally, we compute column (7) by applying the negation table to the values in column (6).

To determine the logical relation for (P ∧ Q) and ¬(¬P ∨ ¬Q) we need to look at columns (5) and (7), since every row under these two columns has the same truth-value. The two formulas are *equivalent*. They are true in exactly the same circumstances. So, *P and Q* means the same as *it is not the case that either P is false or Q is false*.

Example: (P ∧ Q), (P ∨ Q)

	1	2	3	4
0	P	Q	(P ∧ Q)	(P ∨ Q)
1	T	T	T	T
2	T	F	F	T
3	F	T	F	T
4	F	F	F	F

In order to determine the logical relation for (P ∧ Q) and (P ∨ Q) we must look at columns (3) and (4). Since, there is a row where both are true, the row where P is T ad Q is T, the two formulas are *satisfiable*. However, since the two formulas do not have the same value for every row they are not equivalent.

Example: (P → ¬Q), (P ∧ Q)

	1	2	3	4	5
0	P	Q	¬Q	(P ∧ Q)	(P → ¬Q)
1	T	T	F	T	F
2	T	F	T	F	T
3	F	T	F	F	T
4	F	F	T	F	T

In order to determine the logical relation for (P → ¬Q) and (P ∧ Q) we look at columns (4) and (5). Since, there is no row under column (4) and (5) where both have a T, the two formulas are *unsatisfiable*, and since it is not the case that all of the rows are the same there is no logical equivalence between the formulas.

Example: (P → (Q → R), (P ∧ Q) → R

	1	2	3	4	5	6	7
0	P	Q	R	(Q → R)	(P ∧ Q)	(P ∧ Q) → R	(P → (Q → R)
1	T	T	T	T	T	T	T
2	T	T	F	F	T	F	F
3	T	F	T	T	F	T	T
4	T	F	F	T	F	T	T
5	F	T	T	T	F	T	T
6	F	T	F	F	F	T	T
7	F	F	T	T	F	T	T
8	F	F	F	T	F	T	T

In order to determine the logical status of (P → (Q → R) and (P ∧ Q) → R, we must look at columns (6) and (7). Since every row of both columns has the same truth-value, the two formulas are *equivalent*.

The key steps for determining the logical status of a proposition or a group of propositions are the following:

Step 1: create a truth-table with the requisite number of rows and columns.

Step 2: compute the value of every formula on every row, given the initial values provided by atomic statement letters for the row.

Step 3: check for the logical status by checking the relevant rows.

7.3.1 Determining the validity of an argument via truth-tables.

Because an argument is a set of statements, one of which is supported by the others, one can test for the validity of an argument by creating a truth-table for each of the premises and the conclusion. The procedure is exactly the same procedure one employs for testing the logical status of a proposition or a group of propositions. The only difference is that the question one asks, when the complete truth-table is done, concerns validity:

Is there any row where the columns for the premises have a T, and the column for the conclusion has an F?

- If the answer to this question is 'yes', then the argument is *invalid*.
- If the answer to this question is 'no', then the argument is *valid*.

The reasoning is the following. A truth-table analysis of an argument tells you that for every possible combination of truth-values of the atomic statements what the truth-value is of the premises and the conclusion. So, if there is some row where the premises are true and the conclusion false, the truth-table analysis informs us that the row with the true premises and false conclusion is the distribution of truth-values to the atomic statements that makes the premises true and the conclusion false.

Example:

If John wins the Election, then Mary will run for Governor. John did not win the election. So, Mary will not run for Governor.

First, we need to translate the argument into PL using a scheme of abbreviation. The symbol '/' is used to denote that the formula that follows it is the conclusion.

P: John wins the election; Q: Mary will run for Governor.

$P \rightarrow Q, \neg P \,/\, \neg Q$

Second, we need to create a truth-table and compute the value of the compound formulas given the initial values of the atomic statement letters P and Q. In the table below, we label premises with 'Pr' and conclusion with 'C'.

			Pr1	Pr2	C
0	P	Q	$(P \rightarrow Q)$	$\neg P$	$\neg Q$
1	T	T	T	F	F
2	T	F	F	F	T
3	F	T	T	T	F
4	F	F	T	T	T

The definition of a valid argument is one where it is impossible for the premises to be true and the conclusion false. The truth-table above shows us that in the case where P = F, and Q = T, the premises are true, and the conclusion false. As a consequence, the argument is invalid.

A truth-table analysis of an argument allows us to determine validity because validity is a modal concept, a concept involving impossibility and possibility. When an argument is valid it is impossible for the premises to be true, and the conclusion false. What this means is that there is no combination of assigning truth-values to the atomic statements that renders the premises true and the conclusion false after computing the value of the compound statements using the truth-tables for the propositional operators. The truth-table gives us all the possible combinations so we know if there is no row where the premises are true, and the conclusion false, the argument is valid.

Example:

If John wins the election, Mary will run for Governor. John won the election. So, Mary will run for Governor.

P: John wins the election; Q: Mary will run for Governor.

$(P \rightarrow Q), P \,/\, Q$

			Pr1	Pr2	C
0	P	Q	$(P \rightarrow Q)$	P	Q
1	T	T	T	T	F
2	T	F	F	T	T
3	F	T	T	F	T
4	F	F	T	F	F

Since no row has premises that are true and a false conclusion, the argument is valid.

Example:

America will invade Iraq only if Saudi Arabia does not condemn outside involvement. Saudi Arabia will not condemn outside involvement only if France supports America. But, France will not support America. So, American will not invade Iraq.

P: America will invade Iraq; Q: Saudi Arabia condemns outside involvement; R: France supports America.

$P \rightarrow \neg Q, \neg Q \rightarrow R, \neg R \,/\, \neg P$

	1	2	3	4	Pr1	Pr2	Pr3	C
0	P	Q	R	¬Q	(P → ¬Q)	¬Q → R	¬R	¬P
1	T	T	T	F	F	T	F	F
2	T	T	F	F	F	T	T	F
3	T	F	T	T	T	T	F	F
4	T	F	F	T	T	F	T	F
5	F	T	T	F	T	T	F	T
6	F	T	F	F	T	T	T	T
7	F	F	T	T	T	T	F	T
8	F	F	F	T	T	F	T	T

This argument consists of three premises and a conclusion. Each of the three atomic statement letters has a column, and the formula ¬Q is given a column, since it is a sub formula of two premises. First the value of each premise is computed by using the truth-table definition for the appropriate main connective, and the conclusion is computed by applying the negation table to column (1). Since none of the rows have all true premises and a false conclusion the argument is valid.

Chapter 8
Indirect-Table Analysis

8.1 Indirect-table test for validity

Once you come to know the truth-table definitions for the propositional connectives, and learn how to construct a complete truth-table test for validity, it becomes apparent that you need not compute every line of a truth-table in order to determine whether an argument is valid. Given that an argument is valid if and only if it is impossible for the premises to be true and the conclusion false, it is simply needless to complete every row of a truth-table. A row cannot show that an argument is invalid if the conclusion is true on that row. The conclusion must be false in order for it to be possible that the row shows the argument to be invalid.

Example: P → Q, Q / P

			Pr1	Pr2	C
0	P	Q	(P → Q)	Q	P
1	T	T			
2	T	F			
3	F	T	T	T	F
4	F	F	T	F	F

Since the conclusion is simply P, the only columns under C that need to be computed are 3 and 4. Given that an argument is valid if and only if it is impossible for the premises to be true and the conclusion false, only rows 3 and 4 could show the invalidity of the argument. And as we see the row 3 gives us true premises and a false conclusion revealing the invalidity of the argument. By keeping the definition of validity in mind and thinking strategically about how to fill out a truth-table one can eliminate useless steps.

The fact that an argument is invalid only at the rows where the conclusion is false, leads to a shortcut method for testing validity based on a simple strategy.

Consider our example: P → Q, Q / P

Step 1: write down the argument in a table such as below

Pr1	Pr2	C
P → Q	Q	P

Step 2: make the conclusion false

Pr1	Pr2	C
P → Q	Q	P
		F

Step 3: try to make the premises true without being forced to assign both T and F to any single atomic statement or formula

Pr1	Pr2	C
P → Q	Q	P
T	T	F
F T		

Step 4: if forced to assign a different truth-value to the same atomic statement or compound formula, then the argument is valid.

In the case above, we are not forced to make a different assignment to any of the atomic statements. So, the argument is invalid because the row shows how the premises can be true and the conclusion false. Namely, when Q is true and P is false.

Example: P → Q, P / Q

Pr1	Pr2	C
P → Q	P	Q
T F	T	F
F		

We start by making the conclusion false, and immediately making the premises with just 'P' true. As a consequence, we are forced to assign P and Q in the conditional the values T and F respectively. However, the definition of → for the case where there is a true antecedent and false consequent is false. So, the conditional is false.

Looking carefully, we can see that our argument is one in which the attempt to make the premises true once the conclusion has been made false leads to a contradiction. So, the argument is valid.

Example: P → Q, (R ∧ Q) ∨ S / P

Pr1	Pr2	C
P → Q	(R ∧ Q) ∨ S	P
F T	T T	F
T	T T	

In the case above, we need not even compute every single atomic statement letter to find the key ingredients of any row that would be an invalidating row of the argument. We start by making the conclusion P false, which then forces us to set the 'P' in 'P → Q' to F, which subsequently makes the conditional true, since a → is always true when the antecedent is false. Next, we move to the second premise which is a disjunction, and since a disjunction is true just in case at least one disjunct is true, we can simply choose to make S true, since that does not force us to determine any other statement. So, the argument is invalid.

Although the short tables above were relatively simple, this is not always the case. Sometimes, one needs to make multiple rows in order to determine whether an argument is invalid via the short method.

Example: P → Q, Q → R, ¬S ∨ V / V ∧ P

The argument above has 3 premises, and 5 atomic statement letters, P, Q, R, S, and T. If we were to make a complete truth-table for the argument we would need 128 rows to cover all the possible combinations of truth-values to atomic statement letters. So the shortcut method is the way to go:

Pr1	Pr2	Pr3	C
P → Q	Q → R	¬S ∨ V	V ∧ P
			F

Step 1: make the conclusion false. However, since there are three ways a conjunction can be false, we need three rows, not just one. Each of the rows will be a distribution of truth-values that renders the conclusion false. One where both conjuncts are false, one where V = F, and P = T, and one where V = T, and P = F.

Step 2: Compute the rows attempting to make the premises true without assigning both T and F to the same atomic statement.

	Pr1	Pr2	Pr3	C
0	P → Q	Q → R	¬S ∨ V	V ∧ P
1	F T F	F T	TF T F	F F F
2	F T F	F T	T T	T F F
3	T T T	T T T	T F T F	F F T

In row 1 by making both V = F and P = F, we are forced to make V in the third premise false, and P in the first premise false. Since a → is true any time the antecedent is false, premise 1 is already rendered true; so we can assign anything to Q in the first premise. Thinking ahead about the second premise, it is wise to assign F to Q in the first and second premise because a → is also true when both the antecedent and consequent are false, allowing premise 1 to remain true; but also allowing premise 2 to be true, since Q is the antecedent of it. Premise 3 must now be made true since V is false. The other major formula of premise 3 has to be made true to make the ∨ true. Since the other major formula is a ¬, the statement letter needs to be assigned F, S = F, and thus ¬S = T. Now all the premises are true and the conclusion false. Row 2 is much the same as row 1.

However, let us look at row 3. In row 3, the conclusion is made false by assigning false to V and true to P. As a consequence, we are forced to assign true to P in premise 1, and since a → with a true antecedent is true only when the consequent is true as well, we are forced to assign false to Q, rendering Q true. Since Q is true, and Q is the antecedent of premise 2, we must make R true as well in order to make premise 2 true. Since, V = F in premise 3, the compound formula ¬S must be made true, which requires making S false.

All three rows turn out to be rows that invalidate the argument; the important thing to note is that one must recognize that this might not be the case:

Example: P → R, Q → S, P ∨ Q / R ∧ S

	Pr1	Pr2	Pr3	C
0	P → R	Q → S	P ∨ Q	R ∧ S
1	F T F	F T F	F T F	F F F
2	T T T	F T F	T T F	T F F
3	F T F	T T T	F T F	F F T

Step 1: make the conclusion false with a row for each way it can be false.

In the case above, the following happens at row 1. The conclusion is made false because R = F, and S = F. This assignment forces us to make the consequent of premise 1 false, which forces us to make the antecedent true, since a → with a true consequent is only true when the antecedent is true as well. In premise 2, we are forced by the fact that S = F in the conclusion to make the consequent false, which forces us to make Q false as well, for the same reason as premise 2. With Q = F and P = F, though, there is no way to make premise 3 true. A disjunction is true when at least one disjunct is true. So row 1 with R = F and S = F, cannot be the invalidating row, since it leads to an attempt to make a disjunction true with false disjuncts, which is impossible.

However, rows 2 and 3 are profitable. In row 2 the conclusion is false because R = T and S = F. As a consequence, we are forced to assign Q = F in premise 2 to make the → true. Since R = T in premise 1 we have two options, since a conditional with a true consequent is true if either the antecedent is true or false. Thinking ahead, it is smart to choose to make P = T in premise 1, so that in premise 3 we get a true ∨. Since premise 3 already has Q = F, we need to make P = T to get the premise true. Thus, row 2 shows that the argument is invalid.

8.2 Notation

When using the shortcut method for determining validity it is customary in the case where the argument is invalid to specify a row that shows the invalidity by the following method: Once done computing a row, use the notation V(x) = T, or V(x) = F. 'V(x)' means the value of x, where x is a wff. In the case of giving your answer V(x) should always be applied to an atomic statement letter. So, in the example above the invalidating row 2 is represented as: V(P) = T, V(R) = T, V(S) = F, V(Q) = F.

8.3 Indirect-tables for determining logical status

Although the shortcut method is easily explained in the case of testing for validity, it can also be used to determine other pieces of information. In particular, the short cut method can also be used to determine whether a group of propositions is satisfiable. Satisfiability also corresponds to the notion of consistency. A group of statements are consistent when they can be true together. Unsatisfiability corresponds to the notion of inconsistency. A group of propositions that are inconsistent cannot be true together.

In order to test whether a group of propositions is consistent using the shortcut method, one simply attempts to make the entire set of formulas true without assigning any formula both the truth-values T and F.

Step 1: put a T under the main connective of every formula to be tested.

Step 2: attempt to find a distribution of truth-values that maintains the truth of the formulas.

Example: P → Q, ¬P ∨ Q

P → Q	¬P ∨ Q
T	T
F F	T F

Example: P → Q, P ∧ ¬Q

P → Q	P ∧ ¬Q
T	T
T F	T TF
F	

In this case, since it cannot be done we find that the two formulas are not satisfiable. Because there is only one way that a conjunction can be true it is best to start by setting V(P) = T and V(Q) = F, so that V(¬Q) = T, and V(P ∧ ¬Q) = T. However, once that is done we immediately see that V(P → Q) = F. So, since there is only one way the conjunction above can be true, and that way makes the corresponding conditional false, the two formulas are unsatisfiable.

When testing for satisfiability it is important to note that if there are multiple ways in which the formulas can be true one *cannot* conclude upon computing a row where some formula is true and another formula is false that the set of formulas is not satisfiable. One must continue to test for satisfiability until all of the possible truth-making rows have been exhausted.

Example: P → ¬Q, P ∨ Q

0	P → ¬Q	P ∨ Q
1	T F F T	T T T

Notice on row 1 where both V(P) = T and the V(Q) = T, the conditional is false and the disjunction is true. At this stage one may not conclude that the two formulas are satisfiable, since this row only shows that on one distribution of truth-values the two formulas do not come out both true. So, one must continue on to another distribution of truth values.

0	P → ¬Q	P ∨ Q
1	T F F T	T T T
2	T T T F	F T F
3	F T T F	F F F

On row 2 we see that the two formulas can both be true at the same time. When V(P) = T and V(Q) = F both formulas are true. However, if we had done row 3 prior to row 2 we would have discovered another row where both formulas are not true. When V(P) = F and V(Q) = F, the conditional is true, but the disjunction is false.

In order for a set of formulas to be satisfiable it must be the case that there is at least one row where all the formulas are true. So, when one is using the short-cut method one cannot conclude in a case where there are multiple ways in which the formulas can be true, that the set is unsatisfiable when they reach a row where some formula is true and another formula is false.

8.4 Summary of tests via indirect procedure:

Test	**Procedure**	**Results**
Satisfiability/ Consistency	Place a T under the main connective of each of the formulas.	If there is at least one row where every formula has a T under it, then the set of formulas is satisfiable.
Validity	Place a T under every premise, and F under the conclusion.	If there is one row where the premises are true, and the conclusion is false, then the argument is invalid.

Chapter 9

Truth-Tree Analysis

9.1 Foundations

As we saw in the previous section, the shortcut method is in part motivated by the fact that making a complete truth-table for an argument is inefficient. If an argument has more than 6 atomic statement letters it will need at least 128 rows for a complete table. 8 rows are more than enough for most of us. The shortcut method provides us with a way out that is based on the idea that an argument is valid if one cannot make the premises true and the conclusion false at the same time without assigning both T and F to any statement or formula. Whenever one can distribute a truth-value to all the atomic statement letters and make every premise true, and the conclusion false, the distribution shows that the argument is invalid.

In addition to the shortcut method, though, there is a completely different and more efficient way to test for validity. The technique involves drawing so-called "truth-trees". A truth-tree employs simple argument diagramming rules that allow one to construct a tree that can be used to determine the validity of an argument. The rules for building truth-trees allow one to break down compound formulas to atomic statements.

The basic idea of a truth-tree is that an invalid argument is equivalent to the consistency of the premises with the denial of the conclusion, and a valid argument is equivalent to the inconsistency of the premises with the denial of the conclusion. A truth-tree for an argument, thus, is simply a diagram of the possibility of having true premises and a false conclusion. If every branch of a tree closes, then the initial argument is valid. If some branch of a tree is open, then the initial argument is invalid. All closed branches in a tree means that the premises are inconsistent with the denial of the conclusion. An open path in a tree shows how the premises can be consistent with the denial of the conclusion.

The *first step* of truth-tree construction is stacking the premises, and denying the conclusion. This is done because all we are considering is the relation between the premises and the negation of the conclusion.

Example: $P \rightarrow Q, Q \rightarrow S$ / $P \rightarrow S$

$$\begin{array}{c} P \rightarrow Q \\ Q \rightarrow S \\ \neg(P \rightarrow S) \end{array}$$

The *second step* involves applying the rules for truth-tree evaluation and construction. The rules are procedures for breaking down compound formulas into their subparts either as the subpart itself or as a negation of it. In the rules below, ○ and △ work like propositional variables *p* and *q*. They stand for a form of a well formed formula just as the lowercase letters of the meta-language of PL act as variables; for example, △ can stand for a statement letter such as P in one argument and for $P \rightarrow Q$ in another argument. The only rule is that they must stand for the same thing in any single argument. The mark ✓ placed next to a formula is used to signify that one has broken down the formula.

One should apply rules to the premises and to complex formulas until all formulas are broken down into either atomic statement letters or the negation of an atomic statement letter. A tree is basically a diagram of how combinations of truth and falsity can be applied to the atomic statement letters.

9.2 Rules for truth-tree construction

1. Any time a formula occurs in a tree and the negation of the formula (if the formula is a negation, then its affirmation) at some line below it on a connected branch, one places an X underneath the formula, signifying that the branch is closed.

$$\begin{array}{c} \bigcirc \\ \underline{\neg\bigcirc} \\ \text{X} \end{array}$$

2. A formula with two negations applying directly to it can be replaced on a line by the formula itself.

$$\begin{array}{c} \underline{\neg\neg\bigcirc} \\ \bigcirc \end{array}$$

3. A negated conditional is broken down into a single trunk with the affirmation of the antecedent followed by the negation of the consequent.

$$\begin{array}{c} \underline{\checkmark\neg(\bigcirc \rightarrow \triangle)} \\ \bigcirc \\ \neg\triangle \end{array}$$

4. A conditional is broken down into two separate trunks, one trunk with the negation of the antecedent, and one with the affirmation of the consequent.

$$\begin{array}{cc} \multicolumn{2}{c}{\underline{\checkmark(\bigcirc \rightarrow \triangle)}} \\ \neg\bigcirc & \triangle \end{array}$$

5. A conjunction is broken down into a single trunk with the affirmation of both conjuncts stacked.

$$\begin{array}{c} \underline{\checkmark(\bigcirc \wedge \triangle)} \\ \bigcirc \\ \triangle \end{array}$$

6. A disjunction is broken down into two trunks, one trunk for each disjunct.

$$\begin{array}{cc} \multicolumn{2}{c}{\underline{\checkmark(\bigcirc \vee \triangle)}} \\ \bigcirc & \triangle \end{array}$$

7. A biconditional is broken down into two trunks, one trunk has both the affirmation of the antecedent and the consequent, and the second trunk has the negation of both the antecedent and consequent.

✓(○ ≡ △)

○	¬△
△	¬○

8. A negated conjunction is broken down into two trunks; each trunk has a negation of one of the conjuncts.

✓¬(○ ∧ △)

¬○	¬△

9. A negated disjunction is broken down into a single trunk with each disjunct negated.

✓¬(○ ∨ △)

¬○

¬△

10. A negated biconditional is broken down into two trunks, one trunk contains the negation of the antecedent, and the affirmation of the consequent, the other trunk contains the affirmation of the antecedent, and the negation of the consequent.

✓¬(○ ≡ △)

¬○	○
△	¬△

Example: (P → Q), (Q → S) / (P → S)

1.		✓(P → Q)			
2.		✓(Q → S)			
3.		¬(P → S)			
4.		P			
5.		¬S			
6.	¬P			Q	
7.	X		¬Q		S
8.			X		X

Step 1 of tree construction requires us to negate the conclusion. Since (P → S) is the conclusion we write down ¬(P → S), and to break it down further we apply the rule for ¬(○ → △), which gives us lines 4 and 5. Now we must break down our two premises. Starting with (P → Q) we apply the rule for ○ → △, and get line 6. Since ¬P is the negation of P, and P is above it, we close the branch with an X by applying rule 1. At line 7 we also open up the second premise (Q → S) by applying the rule for ○ → △. At line 8 we place an X under both ¬Q and S because each has above it their negation. Since every branch of the tree has an X under it, the argument is valid.

Example: (P → Q), (R → Q), (P ∨ R) / (Q → S)

1.		✓(P → Q)		
2.		✓(R → Q)		
3.		✓(P ∨ R)		
4.		¬(Q → S)		
5.		Q		
6.		¬S		
7.	P		R	
8.	¬P	Q	¬R	Q
9.	X		X	

Step 1 of tree construction tells us to negate the conclusion (Q → S). The negated conditional ¬(Q → S) allows us to apply the rule ¬(○ → △) to give us lines 5 and 6. Starting at line 7 we can break down any of the premises we want to. Thinking ahead breaking down (P ∨ R) first is wise, since it starts two trunks each with the antecedent of one of the other premises. At line 8 we break down the other two premises by using the rule for the conditional. At line 9 we place an X under ¬P and ¬R, since P and R are above them. There are no more premises to break down, or rules to apply to compound formulas; so, since two branches are open the argument is invalid.

Main Principles of Tree Evaluation and Construction

If every branch of a tree closes, the argument is valid.

If at least one branch of a tree is open, then the argument is invalid.

Always break down premises with single trunk construction prior to breaking down premises with multiple trunk construction: conjunctions, negated disjunctions, and negated conditionals come first.

Example: (R ≡ S), (R → Q), (Q ∧ P) / (P ∧ S)

1.		✓$(R \equiv S)$				
2.		✓$(R \rightarrow Q)$				
3.		✓$(Q \wedge P)$				
4.		$\neg(P \wedge S)$				
5.		Q				
6.		P				
7.	$\neg P$				$\neg S$	
8.	X			$\neg R$		$\neg Q$
9.			R		$\neg R$	X
10.			S		$\neg S$	
11.			X			

The argument above is invalid because at line 11 there is an open branch under ¬S. In this case the conclusion was not the first line that was broken down. Rather, line 3 was broken down first because it is the only formula with a single trunk break down. In line 7 the conclusion is broken down. It leads to the immediate closing of a branch in line 8. Some of the branches under ¬S close when the second and first premises are broken down sequentially.

It is also important to note that because there are various ways to break down an argument, depending on the order in which the premises are broken down, an argument may have more than one tree. However, if an argument is valid under one tree it cannot be invalid under another tree. And if an argument is invalid under one tree, it cannot be valid under another tree.

Example: $(P \rightarrow Q)$, $(\neg Q \vee R)$/ $(P \rightarrow R)$

Proof A

1.			$(P \to Q)$	
2.			$(\neg Q \lor R)$	
3.			$\neg(P \to R)$	
4.			P	
5.			$\neg R$	
6.		$\neg Q$		R
7.	$\neg P$		Q	X
8.	X		X	

Proof B

1.		$(P \to Q)$			
2.		$(\neg Q \lor R)$			
3.		$\neg(P \to R)$			
4.		P			
5.		$\neg R$			
6.	$\neg P$			Q	
7.	X		$\neg Q$		R
8.			X		X

Both proof A and B show the argument to be valid because all branches close. The difference between A and B occur at lines after 5. In proof A premise 2 is broken down first in line 6. In proof B, premise 1 is broken down first in line 6.

The rules given so far do not cover how to break down formulas involving only disjunction or conjunction alone, with or without negation.

For example, we do not have rules for
$(P \land Q \land R \land S)$, $(\neg S \lor \neg T \lor \neg V)$, or $\neg(P \land Q \land R)$.

These cases concern multiple conjunctions, disjunctions, and negations of multiple conjunctions or disjunctions.

11. A conjunction of multiple conjuncts is broken down into a single trunk containing all the conjuncts.

✓(○ ∧ △ ∧ □)
○
△
□

12. A disjunction of multiple disjuncts is broken down into n-trunks, one for each of the n-disjunct.

✓(○ ∨ △ ∨ □)
○ △ □

13. A negation of multiple conjuncts is broken down into n-trunks, each trunk being the negation of one of the n-conjuncts.

✓¬(○ ∧ △ ∧ □)
¬○ ¬△ ¬□

14. A negation of disjuncts is broken down into a single trunk of the negation of each disjunct.

✓¬(○ ∨ △ ∨ □)
¬○
¬△
¬□

Example: (P ∨ Q ∨ R), (S → ¬P), (S → ¬Q) / (S → R)

1.			✓$(P \vee Q \vee R)$				
2.			✓$(S \rightarrow \neg P)$				
3.			✓$(S \rightarrow \neg Q)$				
4.			$\neg (S \rightarrow R)$				
5.			S				
6.			$\neg R$				
7.	P		Q				R
8.	$\neg S$	$\neg P$	$\neg S$	$\neg P$			X
9.	X	X	X	$\neg S$	$\neg Q$		
10.				X	X		

The argument is valid because all branches close. Line 4 is the negation of our conclusion $(S \rightarrow R)$. At line 5 and 6 the conclusion is broken down, followed by the break down of premise 1 in line 7. Line 8 is the break down of premise 2, which must go underneath both P and Q since they are open branches. We close the R branch because of the presence of ¬R above it in line 6. Line 9 is the closing of ¬S branch because of the presence of S at line 5, and the closing of ¬P branch because of the presence of P in line 7, However, we can only close ¬S under Q, and not ¬P since it has no P above it. At line 9 premise 3 is broken down. At line 10 ¬S closes because of S in line 5, and ¬Q closes because of Q in line 7.

9.2 Supplement: truth-trees for testing logical relations

Truth-trees can be used not only to test for validity but also to test for consistency. This resembles the way in which indirect truth-tables can be used to test for validity as well as for consistency. In order to test to see whether a set of formulas is consistent one follows a procedure very similar to the procedure used to test if an argument is valid.

To see if a set of formulas is consistent, simply stack the formulas and use the tree construction rules to break down the formulas and construct a tree.

If there is one path that is open, then the set of formulas is consistent.

If no path is open, then the set of formulas is inconsistent.

While the first step in testing whether an argument is valid using the tree-method is to negate the conclusion, in testing for consistency one does not negate any of the initial formulas. The reason why is because tree construction is a consistency test. In testing for validity what we are doing is testing to see whether the premises and the negation of the conclusion are consistent, since if they are consistent it is possible for the premises to be true and the conclusion false. So, in a plain consistency test, we do not negate any of the formulas.

Example:

$$(P \rightarrow Q)$$
$$(\neg P \rightarrow \neg Q)$$

1.	$\neg P$		Q	
2.	$\neg\neg P$	$\neg Q$	$\neg\neg P$	$\neg Q$
3.	X	O	O	X

There are two paths that are open, so the formulas are consistent. At 1 we broke down $(P \rightarrow Q)$ using the rule for $\rightarrow$, and at line 2 we broke down $(\neg P \rightarrow \neg Q)$ using the rule for $\rightarrow$. At line 3 we found two contradictions, and thus closed two branches. The others remain open.

Example:

$\neg(\neg P \lor Q)$
$(P \rightarrow Q)$

1.		$\neg\neg P$	
		$\neg Q$	
2.	$\neg P$		Q
3.	X		X

There are no paths that are open, so the formulas are inconsistent.
At line 1 we broke down $\neg(\neg P \lor Q)$ with the $\neg(\bigcirc \lor \triangle)$ rule. At line 2 we broke down $(P \rightarrow Q)$ with the $\rightarrow$ rule. At line three all paths close because of contradictions.

Unit 4
Formal Techniques of Analysis II

Chapter 10
Natural Deduction – *Rules of Inference*

10.1 Elements

Although truth-table analysis and truth-tree analysis are very important ways of testing for validity, there is yet a further way of testing for validity known as **natural deduction**. If an argument is valid, then it is impossible for the premises to be true and the conclusion false. As a consequence, one should be able to *derive* the conclusion from the premises using valid rules of deduction. Valid rules of deduction are rules that are truth-preserving. A valid rule never takes one from a true premise to a false conclusion. Natural deduction is thus a way of providing a proof of the fact that an argument is valid by showing how one can derive the conclusion from the premises using rules that never move from truth to falsity.

Natural deduction depends heavily on the application of rules of inference. As a consequence, it is important to keep in mind what a *proper substitution instance of an inference pattern* is. In order to apply the rules of the system one must see that a specific formula is a proper substitution instance of a rule. Suppose our rule is the following:

$(p \rightarrow q), p \,/\, q$

All of the following are proper substitution instances of it.

$(\mathrm{P} \rightarrow \mathrm{Q}), \mathrm{P} / \mathrm{Q}$	P for p and Q for q
$(\neg\mathrm{R} \rightarrow \neg\mathrm{S}), \neg\mathrm{R} \,/\, \neg\mathrm{S}$	$\neg$R for p and $\neg$S for q

(T ∧ Q) → (R ∨ V), (T ∧ Q) / (R ∨ V) (T ∧ Q) for *p* and (R ∨ V) for *q*

All of the following are improper substitution instances of it.

(P → (Q → R), P / R   (P → (Q for *p* and R for *q*

(S → T) → P, T / P   (S → T) for *p* and T for *q*, and P for *q*

(S ∨ V) → (Q ∨ R), (S ∨ V), Q / R   (S ∨ V) for *p* and Q for *q*, and R for *q*

There are two main rules that govern substitution:

No propositional variable in a rule can be assigned more than one statement in a given substitution.

More than one propositional variable in a rule can stand for the same statement in a given substitution.

The first rule tells us that *p* cannot be assigned both R and S in a given substitution.

Example: (*p* → *q*), *p* / *q*   (R → T), S / T

Here *q* is assigned T, but *p* is assigned both R and S.

The second rule tells us that *p* and *q* can both be assigned R in a given substitution.

Example: (*p* → *q*), *p* / *q*   (R → R), R / R

Here *p* is assigned R, and *q* is assigned R.

A natural deduction proof of the conclusion of an argument from the premises of the argument is represented by a numbering system and two columns: the left most column lists the formulas, the right most column lists the status of the formula or how it was derived from prior formulas. The right column is called the justification column, since it often contains the justification for a deductive step. When the justification for the placement of a formula on a line involves an inference the justification line cites the lines used, and the name of the rule(s) applied. Our first two inference rules will be introduced alongside our first set of natural deduction proofs.

Let us introduce the idea of a natural deduction proof along with the first two rules of natural deduction.

Modus Ponens and Modus Tollens

Example: (P → Q), (Q → R), P / R

Suppose we wanted to derive the conclusion R, from the premises (P → Q), (Q → R), and P

Step 1: list the premises sequentially with the word 'premise' in the justification column.

1. (P → Q)	Premise
2. (Q → R)	Premise
3. P	Premise

Step 2: apply rules of inference until the conclusion is reached.

Modus Ponens (MP)

(*p* → *q*), *p* / *q*

4. Q	1, 3 MP
5. R	2, 4 MP

Completed Proof:

1. (P → Q)	Premise
2. (Q → R)	Premise

3. P	Premise
4. Q	1, 3, MP
5. R	2, 4, MP

The argument is valid because the conclusion R can be derived by the valid rule of inference (MP) from the premises. The proof only involves two applications of (MP), the justification column for lines 4 and 5 states which lines are being employed in each application of (MP).

Example: $(P \rightarrow Q), (Q \rightarrow R), \neg R / \neg P$

Suppose we wanted to derive the conclusion $\neg P$ from the premises $(P \rightarrow Q), (Q \rightarrow R)$, and $\neg R$

Step 1: list the premises sequentially followed by the word 'premise' in the justification column

1. $(P \rightarrow Q)$	Premise
2. $(Q \rightarrow R)$	Premise
3. $\neg R$	Premise

Step 2: apply rules of inference until the conclusion is reached.

Modus Tollens (MT)

$(p \rightarrow q), \neg q / \neg p$

4. $\neg Q$	2, 3 (MT)
5. $\neg P$	1, 4 (MT)

Completed Proof:

1. $(P \rightarrow Q)$	Premise
2. $(Q \rightarrow R)$	Premise
3. $\neg R$	Premise
4. $\neg Q$	2, 3 (MT)

5. $\neg P$	1, 4 (MT)

The argument is valid because the conclusion $\neg P$ can be derived by the valid rule of inference (MT) from the premises. The proof only involves two applications of (MT), the justification column in lines 4 and 5 states which lines are being used in each application of (MT).

Although the examples above only involve a single rule of inference, most interesting cases involve multiple rules.

Example: $(P \rightarrow Q)$, $(\neg P \rightarrow \neg R)$, $\neg Q$ / $\neg R$

1. $(P \rightarrow Q)$	Premise
2. $(\neg P \rightarrow \neg R)$	Premise
3. $\neg Q$	Premise
4. $\neg P$	1, 3 (MT)
5. $\neg R$	2, 4 (MP)

Here (MT) must be applied first in order to use (MP) to get the conclusion. Proofs involving multiple rules often require one to think ahead as to which rule to use first in order to allow for the application of a rule later.

There are many rules of inference in the system PL. The rules are of two types: rules of inference and rules of replacement. Both (MP) and (MT) are rules of inference and the rules that follow immediately are all rules of inference. We will turn to rules of replacement in 5.2.

Disjunctive Syllogism and Hypothetical Syllogism

Example: $(P \vee (Q \vee R)$, $\neg P$, $\neg Q$ / R

1. $(P \vee (Q \vee R)$	Premise
2. $\neg P$	Premise

3. ¬Q Premise

Disjunctive Syllogism (DS)

$(p \vee q), \neg\, q \,/\, p$

4. (Q ∨ R) 1, 2 (DS)
5. R 3, 4 (DS)

In line 4 (DS) is applied to 1 and 2 where (Q ∨ R) is the *q* disjunct of the argument form ($p \vee q$). Since (Q ∨ R) is a well formed formula it is a substitution instance of *q*.

Completed Proof:

1. (P ∨ (Q ∨ R) Premise
2. ¬P Premise
3. ¬Q Premise
4. (Q ∨ R) 1, 2 (DS)
5. R 3, 4 (DS)

Example: (P → Q), (Q → R) / (P → S)

1. (P → Q) Premise
2. (Q → R) Premise

Hypothetical Syllogism (HS)

$(p \rightarrow q), (q \rightarrow r) \,/\, (p \rightarrow r)$

3. (P → R) 1, 2 (HS)

Completed Proof:

1. (P → Q) Premise
2. (Q → R) Premise
3. (P → R) 1, 2 (HS)

(HS) is a rule of inference that encodes the fact that material conditionals are transitive just like the relations 'greater than', and 'less than'. The proof above is a direct application of (HS) to the premises to derive the conclusion.

Just as before, though, (HS) is often used in proofs involving other rules.

Example: (P → Q) ∨ R), ¬ R, (Q → S) / (P → S)

1. (P → Q) ∨ R)	Premise
2. ¬R	Premise
3. (Q → S)	Premise
4. (P → Q)	1, 2 (DS)
5. (P → S)	3, 4 (HS)

And again, just as before, the order of the application of rules is relevant to deriving the conclusion. One cannot derive the conclusion above unless one first applies (DS).

Example: (Q ∨ R), (Q ∨ R) → (P ∨ (S →T), (T → V) / (S → V)

1. (Q ∨ R)	Premise
2. (Q ∨ R) → (P ∨ (S → T)	Premise
3. ¬P	Premise
4. (T → V)	Premise
5. (P ∨ (S → T)	1, 2 (MP)
6. (S → T)	3, 6 (DS)
7. S → V	4, 6 (HS)

Example: (P → Q), (Q → (S ∨ T), P, ¬S / T

1. (P → Q)	Premise
2. (Q → (S ∨ T)	Premise
3. P	Premise
4. ¬S	Premise
5. (P → (S ∨ T)	1, 2 (HS)
6. (S ∨ T)	3, 5 (MP)

7. T	4, 6 (DS)

In this case, however, (HS) need not be done first:

5. Q	1, 3 (MP)
6. (S ∨ T)	2, 5 (MP)
7. T	4, 6 (DS)

Simplification and Conjunction

There are two simple rules that govern conjunctions. One rule, simplification allows you to break down a conjunction. The other rule, conjunction, allows you to make a conjunction from other formulas.

Simplification (Simp)

(*p* ∧ *q*) / *p*

Conjunction (Con)

p, *q* / (*p* ∧ *q*)

(Simp) allows us to break a conjunction down into its conjuncts. Anytime a conjunction is true, both the conjuncts are true, and so we can break down the conjunction.

(Con) allows us to create a conjunction from two independent statements. If *p* is true and *q* is true, then *p* ∧ *q* is true.

Example: P, Q, (P ∧ Q) → (R ∧ S) / R

1. P	Premise
2. Q	Premise
3. (P ∧ Q) → (R ∧ S)	Premise
4. (P ∧ Q)	1, 2 (Con)

5. $(R \wedge S)$	3, 4 (MP)
6. R	5 (Simp)

Premise 1 and 2 allow for the use of (Con) in creating the antecedent of the conjunction, which in turn allows for a use of (MP) to arrive at the consequent; which finally is simplified to get the conclusion.

Addition and Constructive Dilemma

One of the more interesting rules in PL is the rule of addition, which allows one to add any statement to an already asserted statement. And one of the most useful argument forms is constructive dilemma where one.

Addition (Add)

$p \,/\, (p \vee q)$

Unlike (Simp) where it is generally accepted that a conjunction can be broken down into its conjuncts, it is generally odd to think that a true statement can be disjoined with any other statement. Here is the truth-table that shows the validity of the inference pattern (Add):

p	q	$(p \vee q)$
T	T	T
T	F	T
F	T	T
F	F	F

There is no row where the premises are true and the conclusion false.
As long as p is true, $(p \vee q)$ is true.

Constructive Dilemma (CD)

$(p \vee q), (p \rightarrow r), (q \rightarrow s) \,/\, (r \vee s)$

(CD) allows us to infer from the disjunction of the antecedent of two conditionals the disjunction of their consequents.

Example: P, $(P \rightarrow R)$, $(Q \rightarrow S)$ / $(R \vee S)$

1. P	Premise
2. $(P \rightarrow R)$	Premise
3. $(Q \rightarrow S)$	Premise
4. $(P \vee Q)$	1 (Add)
5. $(R \vee S)$	2, 3, 4 (CD)

Often times (Add) is also used to get the conclusion of an argument.

Example: $(P \rightarrow Q)$, $\neg Q$ / $(\neg P \vee R) \vee S)$

1. $(P \rightarrow Q)$	Premise
2. $\neg Q$	Premise
3. $\neg P$	1, 2 (MT)
4. $(\neg P \vee R)$	3 (Add)
5. $(\neg P \vee R) \vee S)$	4 (Add)

Example: $\neg P$, $(R \rightarrow P)$, $(S \rightarrow Q) \rightarrow T)$, R, $\neg T$ / Q

1. $\neg P$	Premise
2. $(R \rightarrow P)$	Premise
3. $(S \rightarrow Q) \rightarrow T)$	Premise
4. R	Premise
5. $\neg T$	Premise
6. $(S \rightarrow Q)$	3, 5 (MT)
7. $(R \vee S)$	4 (Add)
8. $(P \vee Q)$	2, 6, 7 (CD)
9. Q	1, 8 (DS)

Rules of Inference	
Name	***Rule***
Modus Ponens (MP)	$(p \rightarrow q), p \,/\, q$
Modus Tollens (MT)	$(p \rightarrow q), \neg q \,/\, \neg p$
Hypothetical Syllogism (HS)	$(p \rightarrow q), (q \rightarrow r) \,/\, (p \rightarrow r)$
Disjunctive Syllogism (DS)	$(p \vee q), \neg p \,/\, q$
Addition (Add)	$p \,/\, (p \vee q)$
Conjunction (Con)	$p, q \,/\, (p \wedge q)$
Simplification (Simp)	$(p \wedge q) \,/\, p$
Constructive Dilemma (CD)	$(p \vee q), (p \rightarrow \mathrm{r}), (q \rightarrow s) \,/\, (r \vee s)$

Ch 11
Natural Deduction – *Rules of Replacement*

11.1 Rules of Replacement

Rules of inference are uni-directional rules. For example, (MP) tells us that it is valid to infer from q from $(p \rightarrow q)$ and p. However, it does not tell us that from q we may infer p or $(p \rightarrow q)$. Likewise, (Simp) allows us to infer from $(p \wedge q)$, p; but it does not allow us to infer from p the conjunction $(p \wedge q)$. In fact every rule of inference is such that it is valid to infer the conclusion from the premises of the argument form, but invalid to infer the premises from the conclusion of the argument form.

In addition to the rules of inference PL has several rules of replacement. Rules of replacement differ from rules of inference in that they are bi-directional rules. These rules allow one to replace a formula with another formula that is logically equivalent to it. Every rule of replacement consists of formulas that are logically equivalent to each other (i.e. have the exact same truth value for the main connective of each for every row of the truth-table). Rules of replacement are very useful for completing complex proofs because they allow one to rearrange and substitute statement letters and formulas so as to apply a rule of inference. '::' will be used to mark that the rule is bi-directional.

Double Negation and DeMorgan's Laws

Double Negation (DN)

$\neg\neg\, p :: p$

If the formula p occurs on any line of a proof one may in a subsequent line replace it with $\neg\neg\, p$ with the justification (DN) because $\neg\neg\, p$ is true just in case p is true.

Example: $(P \rightarrow Q), \neg\neg P$ / Q

1. $(P \rightarrow Q)$	Premise
2. $\neg\neg P$	Premise
3. P	2 (DN)
4. Q	1, 3 (MP)

Here (DN) is used to set up an application of (MP). (MP) cannot be done on $(P \rightarrow Q)$ and $\neg\neg P$, given that the antecedent of the conditional is not of the same form.

In addition to the utility that rules of replacement give to setting up the application of rules of inference and aiding in the construction of proofs, in and of themselves they are quite interesting and revealing about the semantics of truth-functional connectives. De Morgan's laws provide an obvious case.

DeMorgan's Laws (DM)

$\neg (p \wedge q) :: (\neg p \vee \neg q)$
$\neg (p \vee q) :: (\neg p \wedge \neg q)$

(DM) tells us that the negation of a conjunction is equivalent to the negation of its conjuncts disjoined, and that the negation of a disjunction is equivalent to the negation of its disjuncts conjoined.

"It is not the case that Fred is tall and Mary is short." means the same as "Either it is not the case Fred is tall or it is not the case Mary is short."

and

"It is not the case that either Fred is tall or Mary is short." means the same as "It is not the case that Fred is tall and it is not the case that Mary is short."

(DM) is an important rule of replacement and it is employed in tandem with (DN).

Example: $(\neg P \wedge Q) / \neg(P \vee \neg Q)$

1. $(\neg P \wedge Q)$	Premise
2. $(\neg P \wedge \neg\neg Q)$	1 (DN)
3. $\neg(P \vee \neg Q)$	2 (DM)

In order to apply (DM) to 1 we must first apply (DN) to the second conjunct of the conjunction so that it has a negation in it, the negation is required to move from $(\neg p \wedge \neg q)$ to $\neg(p \vee q)$. Then when we apply (DM) a negation remains in front of Q.

A truth-table can be used to show that the argument is valid:

0	P	Q	$\neg P$	$\neg Q$	$\neg P \wedge Q$	$(P \vee \neg Q)$	$\neg(P \vee \neg Q)$
1	T	T	F	F	F	T	F
2	T	F	F	T	F	T	F
3	F	T	T	F	T	F	T
4	F	F	T	T	F	T	F

The formulas to be compared are shaded. Notice that for every row the main connective in each formula has the same truth-value. This means that the two formulas are equivalent. So, the inference from the former to the latter is valid.

Example: $\neg(\neg(Q \wedge R) \vee P) / (Q \wedge R)$

1. $\neg(\neg(Q \wedge R) \vee P)$	Premise
2. $\neg\neg(Q \wedge R) \wedge \neg P$	1 (DM)
3. $(Q \wedge R) \wedge \neg P)$	2 (DN)
4. $(Q \wedge R)$	3 (Simp)

Here we apply (DM) first in line 2. There is no need to use (DN) first. We can simply take $\neg(Q \wedge R)$ as a substitution instance of q in the form $\neg(p \vee q)$. After (DM) we use (DN) to eliminate the negations, and use (Simp) to get the conclusion.

Because of the involvement of negation with disjunction and conjunction in (DM) there are multiple cases of transformation. For the sake of explicitness the table below lists the variations with the rule (DN) applied if applicable.

De Morgan's Equivalences	
$p \wedge q$	$\neg(\neg p \vee \neg q)$
$p \vee q$	$\neg(\neg p \wedge \neg q)$
$\neg(p \wedge q)$	$(\neg p \vee \neg q)$
$\neg(p \vee q)$	$(\neg p \wedge \neg q)$
$\neg p \wedge q$	$\neg(p \vee \neg q)$
$p \wedge \neg q$	$\neg(\neg p \vee q)$
$\neg p \vee q$	$\neg(p \wedge \neg q)$
$p \vee \neg q$	$\neg(\neg p \wedge q)$
$\neg(\neg p \wedge q)$	$p \vee \neg q$
$\neg(p \wedge \neg q)$	$\neg p \vee q$
$\neg(\neg p \vee q)$	$p \wedge \neg q$
$\neg(p \vee \neg q)$	$\neg p \wedge q$

The most important thing about applying (DM) is determining the main connective of the formula to which you want to apply it, and then keeping track of negations. However, there is another secret to applying (DM), and that is to simply remember that a disjunction is false only when both disjuncts are false, and a conjunction is true when both conjuncts are true. Keeping that in mind will guide you in converting between conjunctions and disjunctions, negated or not.

Commutation, Association, and Distribution

As discussed before conjunctions and disjunctions are commutative, associative, and distributive. The system PL recognizes this fact through rules of replacement.

Commutation (Com)

$(p \vee q) :: (q \vee p)$

$(p \wedge q) :: (q \wedge p)$

Association (Asc)

$(p \vee (q \vee r) :: (p \vee q) \vee r)$
$(p \wedge (q \wedge r) :: (p \wedge q) \wedge r)$

Distribution (Dist)

$(p \wedge (q \vee r) :: (p \wedge q) \vee (p \wedge r)$
$(p \vee (q \wedge r) :: (p \vee q) \wedge (p \vee r)$

These rules are very useful for setting up the application of a rule of inference.

Example: (P ∧ Q), ¬Q / R

1. (P ∧ Q)	Premise
2. ¬Q	Premise
3. (Q ∧ P)	2 (Com)
4. Q	3 (Simp)
5. (Q ∨ R)	4 (Add)
5. R	1, 5 (DS)

In this proof simplification cannot be performed until commutation is performed on the conjunction.

Example: ((P ∨ Q) ∧ (P ∨ R)) → (S ∨ T), P / (S ∨ T)

1. ((P ∨ Q) ∧ (P ∨ R)) → (S ∨ T)	Premise
2. P	Premise
3. (P ∨ (Q ∧ R)	2 (Add)
4. (P ∨ Q) ∧ (P ∨ R)	3 (Dist)
5. (S ∨ T)	1, 4 (MP)

In this case we have a complex antecedent in the first premise. Notice that the conclusion is simply the consequent of the conditional. So, (MP) would be an ideal way to get the conclusion. However, our only other premise is P. As a consequence, we need to use addition to add the conjunction that we will then distribute in order to get the consequent. We add (Q ∧ R), since that distributes out to the correct conjunction (P ∨ Q) ∧ (P ∨ R).

Association can often be used to perform (DS) or (Simp).

Example: (P ∨ (Q ∨ R), (P ∨ Q) → S), ¬R / S

1. (P ∨ (Q ∨ R)	Premise
2. (P ∨ Q) → S)	Premise
3. ¬R	Premise
4. (P ∨ Q) ∨ R)	1 (Asc)
5. (R ∨ (P ∨ Q)	4 (Com)
6. (P ∨ Q)	3, 5 (DS)
7. S	2, 6 (MP)

Here we perform (Asc) and (Com) first to set up a (DS), and then use (MP) to derive the conclusion.

Contraposition, Implication, and Exportation

There are three rules of replacement governing the material conditional that most systems of natural deduction recognize.

Contraposition (Cont)

$(p \rightarrow q) :: (\neg q \rightarrow \neg p)$

Contraposition is an intuitive relation governing the material conditional. If a material conditional expresses the idea that the consequent is a necessary condition for the antecedent, then it follows that the falsity of the consequent implies the falsity of the antecedent. "France advances to the quarter finals only if Germany loses" is equivalent to if Germany does not lose, then France will not advance to the quarter finals." (Cont) is also very useful in proofs. Often one needs to perform (Cont) in order to apply (MP), (MT), or (HS).

Example: (P → Q), (¬P → ¬S), S / Q

Version A

1. (P → Q)	Premise
2. (¬P → ¬S)	Premise
3. S	Premise
4. (¬Q → ¬P)	1 (Cont)
5. (¬Q → ¬S)	2, 4 (HS)
6. ¬¬S	3 (DN)
7. ¬¬Q	5, 6 (MT)
8. Q	7, (DN)

Of course the proof also could have been done using (Cont) to switch around premise 2 to set it up for a (HS). This in turn allows for the use of (MP) rather than (MT) to arrive at the conclusion.

Version B

1. (P → Q)	Premise
2. (¬P → ¬S)	Premise
3. S	Premise
4. (S → P)	2 (Cont)
5. (S → Q)	1, 4 (HS)
6. Q	3, 5 (MP)

Implication (Imp)

$(p \rightarrow q) :: (\neg p \vee q)$

Implication connects us to the semantics of the material conditional. A material conditional is true just in case the antecedent is false or the consequent is true

0	p	q	$(p \rightarrow q)$
1	T	T	T
2	T	F	F
3	F	T	T
4	F	F	T

In addition, by way of applying (DM) to $\neg p \vee q$ the conditional is connected to another equivalence involving negation and conjunction.

0	p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \vee q$	$\neg(p \wedge \neg q)$
1	T	T	F	F	T	T	T F
2	T	F	F	T	F	F	F T
3	F	T	T	F	T	T	T F
4	F	F	T	T	T	T	T T

Exportation (Exp)

$(p \rightarrow (q \rightarrow r) :: (p \wedge q) \rightarrow r)$

(Exp) is a simple form of reasoning. It captures the idea that a conditional that implies a conditional is equivalent to the conjunction of the antecedents implying the consequent of the embedded conditional.

Example: $(P \rightarrow (\neg Q \vee R), (P \wedge Q) / R$

1. $(P \rightarrow (\neg Q \vee R)$	Premise
2. $(P \wedge Q)$	Premise
3. $(P \rightarrow (Q \rightarrow R)$	1 (Imp)
4. $(P \wedge Q) \rightarrow R)$	3 (Exp)
5. R	2, 4 (MP)

The use of exportation and implication greatly shortens the proof. Without (Exp) one would have to derive it as follows.

1. $(P \rightarrow (\neg Q \vee R)$	Premise
2. $(P \wedge Q)$	Premise
3. P	2, (Simp)
4. $(\neg Q \vee R)$	1, 3 MP
5. $(Q \wedge P)$	2 (Com)
6. Q	5 (Simp)
7. $\neg\neg Q$	6 (DN)
8. R	4, 7 (DS)

Tautology and Equivalence

Sometimes in the course of a proof it is necessary to introduce or eliminate redundant statements. The rule Tautology is used primarily for this purpose.

Tautology (Taut)

$p :: (p \wedge p)$
$p :: (p \vee p)$

Anytime a statement is true, the conjunction containing it as both conjuncts is true, and if the statement is false, the conjunction is false. Anytime a statement is true, the disjunction containing it as both disjuncts is true, and if the statement is false, the disjunction is false.

Example: $(P \rightarrow \neg P) / \neg P$

1. $(P \rightarrow \neg P)$	Premise
2. $(\neg P \vee \neg P)$	1 (Imp)
3. $\neg P$	2 (Taut)

Finally, there are two rules of replacement governing biconditionals. These rules of replacement capture what a biconditional is and what its truth conditions are.

Equivalence (Equiv)

$(p \equiv q) :: (p \rightarrow q) \wedge (q \rightarrow p)$
$(p \equiv q) :: (p \wedge q) \vee (\neg p \wedge \neg q)$

The first equivalence captures that a biconditional is simply a conjunction of two conditionals. The first conjunct of the biconditional is often called the 'left-to-right direction', the second conjunct the 'right-to-left direction'. The second equivalence captures the fact that a biconditional is true either when both statements are true or both are false. (Equiv) is often important for proofs. Here is a classic case.

Example: $(P \equiv Q), (Q \rightarrow R), P \,/\, R$

1. $(P \equiv Q)$	Premise
2. $(Q \rightarrow R)$	Premise
3. P	Premise
4. $(P \rightarrow Q) \wedge (Q \rightarrow P)$	1 (Equiv)
5. $(P \rightarrow Q)$	4 (Simp)
6. Q	3, 5 (MP)
7. R	2, 6 (MP)

In order to perform (MP) to get the conclusion one must first apply (Equiv) to break down the biconditional into the two conditionals that make it up. It is often important to remember that a biconditional is simply the conjunction of two conditionals, and as a result an easy way to deal with it is simply to break it down into the two conditionals.

11.2 Proof Strategies

11.2.1 Compressing a Proof

Consider the following proof

1. $(P \wedge Q)$	Premise
2. $(Q \wedge P)$	1 (Com)
3. Q	2 (Simp)

Strictly speaking this is the correct and explicit deduction of Q from $(P \wedge Q)$. However the following is more efficient

1. $(P \wedge Q)$	
2. Q	1, (Com), (Simp)

Here the justification line shows that two rules were applied at once to a line. This procedure is known as collapsing steps or compressing a proof. Rather than having separate numbered lines for a deductive step, one applies multiple moves at once, and cites the moves in the justification line. This is very common given that certain combinations of rules are easy to read together.
Consider the following proofs

Version A

1. $(P \rightarrow Q)$	Premise
2. $\neg\neg P$	Premise
3. P	2 (DN)
4. Q	1, 3 (MP)

Version B

1. $(P \rightarrow Q)$	Premise
2. $\neg\neg P$	Premise
3. Q	1, 2, (DN), (MP)

Version B contains less steps and is just as clear to one who knows the rules as version A. So, collapsing steps is often useful and efficient. (Com) and (DN) often are performed at the same time with other rules. However, even though it is true that a great many proofs could be reduced to a single line proof by performing all of the steps at once, it would ruin the point of making a proof to do so.

Example: $(P \wedge (Q \vee R), \neg(Q \wedge R) / (P \wedge Q)$

1. $(P \wedge (Q \vee R)$	Premise
2. $\neg(Q \wedge R)$	Premise
3. $(P \wedge Q)$	1, 2, (Dist), (DS)

Here (Dist) and (DS) have been done together. But this is a case in which collapsing should not be done. Distribution can often be done improperly, and as a consequence one should perform it on a separate line. By contrast, (Com) and (DN) are so basic that there is little room for error.

There is no explicit rule covering when steps should be collapsed. In general, one should aim to make their proof understandable to others. If collapsing steps impedes the ability of the intended audience to understand a proof, one should not collapse steps.

11.2.2 Techniques for Proving

One of the toughest, yet exciting things, about natural deduction is coming up with a strategy for deriving the conclusion. As we have already seen in some cases rules have to be applied in a specific order to derive the conclusion; and in other cases the choice of rules can greatly shorten a proof. The key to constructing a proof is to think about what the conclusion is, and how rules can be used to derive it. Remember that proving something by natural deduction is a creative process.

Here are some general proof strategies guidelines—

- Look for statement portions that are repeated with a view toward applying (DS), (MP), (HS), and (MT).
- Small statements, such as P or ¬Q are extremely useful, since they are often used in rules, such as (MP), (DS), and (MT).
- Work backwards from the conclusion.
- Look for new information by employing unused premises.
- When the conclusion contains atomic statements not found in any premise, use (Add) to introduce this information.
- Break down complex sentences with (DM), (Simp), and (Equiv).
- Use (Imp) when you have a mix of conditionals and disjunctions.

Chapter 12

Natural Deduction – *Proof Rules*

12.0 The general idea of proof rules

Unlike rules of inference, which are uni-directional and atomic, taking only single set of premises and yielding a single conclusion in a given instance, and unlike rules of replacement that are bi-directional and can apply to parts of formulas, proof rules are whole strategies for proving specific kinds of statements. The two most commonly discussed proof rules are: conditional proof and indirect proof. The point of conditional proof is to provide one with a strategy for proving a conditional. The point of indirect proof is to provide one with a strategy for proving a statement by exploring the consequence of assuming that its negation is true.

12.1 Conditional Proof

The rule of conditional proof maintains the following

If at any stage in a proof you want to or need to prove $(p \rightarrow q)$, if you assume that p is true, and can by valid steps of employing only prior lines, arrive at q, then you have proven $(p \rightarrow q)$.

Example: P, (P → ¬Q), (Q ∨ S) / (P → S)

1. P	Premise
2. P → ¬Q	Premise
3. (Q ∨ S)	Premise
4. P	Ass (CP)
5. ¬Q	4, 5 (MP)
6. S	3,4 (DS)
7. (P → S)	4-6 (CP)

In this proof we open by Ass (CP), which stands for *assumption for conditional proof*. We assume the antecedent of the conditional we want to prove: $(P \rightarrow S)$. Next we use the information gained by the assumption, P, as well as the other premises to arrive at S, on a line by itself. Once that has occurred we can on the next line write $(P \rightarrow S)$ with the justification of (CP) for conditional proof.

In general the idea behind conditional proof is that if we assume the antecedent of a conditional and then use only information we already know to be true as well as only truth-preserving rules of inference and arrive at another formula we have proven the conditional joining them. That is in general on the assumption that arbitrary p is true, if we arrive through valid reasoning at q, we have shown $(p \rightarrow q)$.

Example: $(P \wedge \neg Q)$, $(R \rightarrow S)$, $(P \wedge \neg R) \rightarrow Z)$ / $(\neg S \rightarrow Z)$

1. $(P \wedge \neg Q)$	Premise
2. $(R \rightarrow S)$	Premise
3. $(P\wedge \neg R) \rightarrow Z)$	Premise
5. $\neg S$	Ass (CP)
6. $\neg R$	2, 3 (MT)
7. P	1 (Simp)
8. $(P \wedge \neg R)$	6, 7 (Con)
9. Z	3, 8 (MP)
10. $(\neg S \rightarrow Z)$	5-9 (CP)

At 5 we assume $\neg S$ for conditional proof since the conclusion is a conditional with $\neg S$ as antecedent. Next we use this new information along with the prior premises to get to Z. At line 6 we apply (MT) to get $\neg R$, which will be used with P to form the antecedent of $(P \wedge \neg R) \rightarrow Z)$. We get P by (Simp) from 1. And we get Z from (MP) on 3 and 8. We have reached Z, so we can at 10 introduce the conditional by (CP).

Example: $(P \vee (Q \vee R)) \rightarrow (S \rightarrow R)$, $(Z \rightarrow (Q \vee (R \vee P)$ / $(Z \wedge S) \rightarrow R)$

1. $(P \vee (Q \vee R)) \rightarrow (S \rightarrow R)$	Premise
2. $(Z \rightarrow (Q \vee (R \vee P)$	Premise
3. $(Z \wedge S)$	Ass (CP)
4. Z	3, (Simp)
5. S	3, (Com),(Simp)
6. $(Q \vee (R \vee P)$	2,4 (MP)
7. $(P \vee (Q \vee R)$	6, (Asc), (Com)
8. $(S \rightarrow R)$	1, 7 (MP)
9. R	5, 8 (MP)
10. $(Z \wedge S) \rightarrow R)$	3-9 (CP)

Example: $(R \vee (S \wedge \neg T)$, $(R \vee S) \rightarrow (U \vee \neg T)$ / $(T \rightarrow U)$

1. $(R \vee (S \wedge \neg T)$	Premise
2. $(R \vee S) \rightarrow (U \vee \neg T)$	Premise
3. T	Ass (CP)
4. $(R \vee S) \wedge (R \vee \neg T)$	1, (Dist)
5. $(R \vee S)$	4, (Simp)
6. $(R \vee \neg T)$	4, (Com),(Simp)
7. $(U \vee \neg T)$	2, 5 (MP)
8. U	3, 7 (Com),(DS)
9. $(T \rightarrow U)$	3-8 (CP)

Example: $(P \rightarrow \neg Q)$, $\neg(R \wedge \neg P)$ / $(R \rightarrow \neg Q)$

1. $(P \rightarrow \neg Q)$	Premise
2. $\neg(R \wedge \neg P)$	Premise
3. R	Ass (CP)
4. $(\neg R \vee P)$	2, (DM), (DN)
5. P	3, 4 (DS)
6. $\neg Q$	1, 5 (MP)
7. $(R \rightarrow \neg Q)$	3-6 (CP)

12.2 Indirect Proof

Unlike conditional proof, which can only be used to prove a conditional, indirect proof is a strategy for proving any formula. The general idea behind indirect proof is the following. Given that either a statement p or its negation, $\neg p$ is true in a bivalent logic, we can assume the opposite of what we want to prove, and if we reach a contradiction, we can infer the original statement.

Suppose we want to prove p

Step 1: assume $\neg p$.

Step 2: using $\neg p$ attempt to derive a contradiction of the form $(r \wedge \neg r)$.

Step 3: if the assumption that $\neg p$ is true leads to a contradiction $(r \wedge \neg r)$, infer that p is true.

Assumption for indirect proof is cited as 'Ass (IP)'.

Example: $(P \rightarrow \neg Q), \neg(R \wedge \neg P)$ / $(R \rightarrow \neg Q)$

1. $(P \rightarrow \neg Q)$	Premise
2. $\neg(R \wedge \neg P)$	Premise
3. $\neg(R \rightarrow \neg Q)$	Ass (IP)
4. $\neg(\neg R \vee \neg Q)$	3, (Imp)
5. $(R \wedge Q)$	4, (DM), (DN)
6. R	5, (Simp)
7. Q	5, (Com),(Simp)
8. $(\neg R \vee P)$	2, (DM), (DN)
9. $(R \rightarrow P)$	8, (Imp)
10. $(R \rightarrow \neg Q)$	1, 9 (HS)
11. $\neg Q$	6, 10 (MP)
12. $(Q \wedge \neg Q)$	7, 11 (Con)
13. $(R \rightarrow \neg Q)$	3-12 (IP)

At 3 we begin with our assumption for indirect proof, which is the negation of the conditional we actually want to prove, $\neg(R \rightarrow \neg Q)$. In the lines there after we attempt to derive a contradiction. Eventually one is found with respect to the value of Q. We explicitly state the contradiction as $(Q \wedge \neg Q)$, which is an instance of the form $(r \wedge \neg r)$. A typical strategy for (IP) when the conclusion is a conditional is to do the following steps immediately. First, perform implication on the conditional, then (DM) and double negation. This should leave you with a conjunction that can be further simplified for further information to use.

Example: $(P \vee Q) \rightarrow (R \wedge S)$, $(S \vee T) \rightarrow W)$, $(P \vee S)$ / W

1. $(P \vee Q) \rightarrow (R \wedge S)$	Premise
2. $(S \vee T) \rightarrow W)$	Premise
3. $(P \vee S)$	Premise
4. $\neg W$	Ass (IP)
5. $\neg(S \vee T)$	2,4 (MT)
6. $(\neg S \wedge \neg T)$	5, (DM)
7. $\neg S$	6, (Simp)
8. $\neg T$	6, (Com),(Simp)
9. P	3, 7 (DS)
10. $(P \vee Q)$	9, (Add)
11. $(R \wedge S)$	1, 10 (MP)
12. S	11, (Com),(Simp)
13. $(S \wedge \neg S)$	7, 12 (Con)
14. W	4-13 (IP)

Example: $(P \vee (Q \wedge R)$, $(P \rightarrow R)$ / R

1. $(P \vee (Q \wedge R)$	Premise
2. $(P \rightarrow R)$	Premise
3. $\neg R$	Ass (IP)
4. $\neg P$	2, 3 (MT)
5. $(Q \wedge R)$	1, 4 (DS)
6. R	5, (Com),(Simp)
7. $(R \wedge \neg R)$	3, 6 (Con)
8. R	3-7 (IP)

In the proof immediately above we have an interesting instance in which on the assumption that the conclusion R is false, we actually reach R within the assumption. In this case we may not stop the proof at line 6. Rather we must first show that ¬R leads to the contradiction (R ∧ ¬R), and from this it follows that R. At times and indirect proof may lead to a contradiction with the very term that is the conclusion. In such cases one must first show the contradiction before drawing the conclusion correctly.

Example: (P → (Q → R), (Q → (R → ¬Q) / (¬P ∨ ¬Q)

1. (P → (Q → R)	Premise
2. (Q → (R → ¬Q)	Premise
3. ¬(¬P ∨ ¬Q)	Ass (IP)
4. (P ∧ Q)	3, (DM), (DN)
5. P	4, (Simp)
6. Q	4, (Com),(Simp)
7. (Q → R)	1, 5 (MP)
8. R	6, 7 (MP)
9. (R → ¬Q)	2, 6 (MP)
10. ¬Q	8,9 (MP)
11. (Q ∧ ¬Q)	6, 7 (Con)
12. (¬P ∨ ¬Q)	3-11 (IP)

Example: (P ∨ Q) → ¬P) / ¬P

1. (P ∨ Q) → ¬P)	Premise
2. ¬¬P	Ass (IP)
3. ¬(P ∨ Q)	1, 2 (MT)
4. (¬P ∧ ¬Q)	3, (DM)
5. ¬P	4, (Simp)
6. (P ∧ ¬P)	2, 5 (DN), (Con)
7. ¬P	2-6 (IP)

12.3 General remarks on proof rules

In propositional logic natural deduction, indirect proof is the strongest proof technique. Any valid argument of propositional logic is such that the conclusion can be derived from the premises by indirect proof. Conditional proof is not as strong as indirect proof it can only prove conditionals.

Concerning length there are no exact guidelines. Sometimes doing a proof simply by using the all of the rules other than (CP) and (IP) will be shorter than using either (CP) or (IP). Other times the opposite will hold. In general, one can always choose to do (IP), although the other techniques are good to use.

In order to develop your familiarity with the different rules be sure to try proofs by different techniques and to explore using different rules.

Guides and Exercises

CH 1: Identifying Arguments

In the passages that follow determine whether an argument is present. If an argument is not present, determine whether the passage is an explanation, an illustration, or a report. Circle all premise and conclusion indicators.

1. The sky appears blue to the human eye because of the way the earth's atmosphere scatters light into waves of a particular length in the color spectrum.
2. There are much more defining characteristics of mammals than having hair and mammary glands. Mammals have large cerebral cortexes, their lower jaw is made of a single bone, and they have three middle-ear bones.
3. If a Democrat wins the election, taxes will increase. It is almost certain that a Democrat will win the election. Therefore, it is almost certain that taxes will increase.
4. "The reason why crime rate increased last year is because funding to law enforcement was cut," said Chief of Police Jed Richards.
5. We should pass legislation that bans the production of foie gras. It is a cruel process in which geese are force fed until their liver swells to a painful size. And any process which causes such harm to another creature for the simple and unnecessary enjoyment of another is wrong.
6. There are many harmful effects that excess amounts of sodium in one's diet can have on the body. It can lead to high blood pressure, kidney failure and an increased chance of stomach cancer.
7. Anti-aging creams are made to make people's skin look younger than it actually is. The only real reason why someone would want their skin to look younger than it really is is because the older one's skin looks, the more one is reminded of their own mortality.

Therefore, the real reason why people buy anti-aging creams is because they do not want to be reminded of their own mortality.

8. I have heard that the cheapest, most effective way to remove grease from bicycle and automotive parts is with kerosene.
9. Often times, deaths due to hypothermia are immediately preceded by feelings of warmth and a desire to remove one's clothes. This is because extremely cold temperatures can cause the hypothalamus to malfunction, and also the muscles which contract the outer blood vessels stop contracting which in turn leads to a quick flood of blood all over the body, making the victim feel warm when he/she is actually freezing to death.
10. Either the murderer was actually present during the time of death of the victim or they were elsewhere when the victim died. The only person who was present during the time of death of the victim was the maid, who we have already proven to not be the murderer. Therefore, we can conclude that the murderer was not present during the time of death of the victim.

CH 2: Diagramming Arguments

Diagram the following arguments.

1. If John runs for mayor, then so will Mary. John will run for mayor. Therefore, so will Mary.
2. Smoking is bad for your health. It makes lung disease more likely. It also ruins your teeth.
3. Exercise helps prevent high blood pressure, it reduces the risk of various kinds of cancer, and it helps control body weight. Apparently exercise is good for your health.
4. If John asks Mary to the prom, then either Justin or Carlos will ask Aurora to the prom. John will ask Mary to the prom. So, either Justin or Carlos will ask Aurora to the prom. But Justin will not ask Aurora to the prom. So, Carlos will ask Aurora to the prom.
5. Osama bin Laden is living either in Pakistan or Iran. He cannot be living in Iran because an anonymous informant said he saw bin Laden leaving his hide-out in Iran and heading for the Pakistani border. So, bin Laden must be living in Pakistan.
6. All politicians are greedy. Mike Pankrast is a politician. So, Mike Pankrast is greedy. All greedy people are bad people. So, Mike Pankrast is a bad person.
7. The possession of firearms should be illegal. Allowing the ownership of firearms has always and will always raise the incidents of gun-related deaths. The reason why this is the case is because when firearms have a widespread legal distribution, there is a greater chance that people will irresponsibly use them in situation in which they feel threatened. It's just human nature to behave this way.
8. If we elect candidate Julie Olsen as mayor our crime rate will go up, because she plans to cut law-enforcement funding and because she plans to pass a city ordinance that would make it

illegal to loiter downtown. So, we shouldn't elect candidate Julie Brown as mayor, because we don't want our crime rate to go up.

9. If there is to be a more globally peaceful state of being for humanity, there must be less violence. In order for there to be less violence, there must less starvation and malnutrition due to economic oppression. In order for there to be less starvation and malnutrition due to economic oppression, the people, governments and corporations in more affluent countries must be willing to abdicate some of the privileges they enjoy due to an inequitable distribution of financial power in the international economy. But the people, governments and corporations in more affluent countries are not willing to abdicate any of the privileges they enjoy due to an inequitable distribution of financial power in the international economy. Therefore, there will not be a more globally peaceful state of being for humanity.
10. The murderer of Dr. Thornburg must be Ms. Davenport. The only three possible suspects were Professor Gunther, Mr. Ivanovich, and Ms. Davenport, and it could not possibly have been either Professor Gunther or Mr. Ivonovich. Professor Gunther was infirm in the hospital during the murder. Furthermore, she had no motive. And Mr. Ivanovich was in Hawaii during the time of the murder. Besides, Dr. Thornburg was shot with a .45 caliber hand gun, and that same hand gun was found in Ms. Davenport's car with her fingerprints on it.

Fallacy Title Guide

Fallacies of Relevance:

Appeal to Force

Appeal to Pity

Appeal to the People

Argument Against the Person

Accident

Straw Man

Missing the Point

Red Herring

Fallacies of Weak Induction

Appeal to Unqualified Authority

Appeal to Ignorance

Hasty Generalization

False Cause

Slippery Slope

Weak Analogy

Fallacies of Presumption, Ambiguity, and Grammatical Analogy

Begging the Question

Complex Question	
False Dichotomy	Amphiboly
Suppressed Evidence	Composition
Equivocation	Division

CH 3: Fallacious Reasoning in Argumentation.
For each passage determine first if there is an argument is present. If there is no argument state what kind of passage is present. Second, if an argument is present determine whether any fallacy is present.

1. Each individual musician in the symphony sounds very good. Therefore, the entire symphony should sound very good.
2. The governor is trying to pass a bill that would switch a percentage of our energy source from coal to solar. But the governor has stock in solar energy, and is probably just trying to pass the bill to increase his assets. Therefore we should oppose the bill.
3. Coca-cola is the best soda in the entire world. This is obvious because everyone drinks it, and the reason they drink it is because it is the best.
4. The company should not have fired Thomas. Even though he is regularly late and he has stolen from the company several times, he has a family to take care of. He can't afford to lose his job right now.
5. The past two times I have been to this restaurant, I have had to wait a while to get my food. The management here must not care at all about quality service.
6. I have looked through both of my coat pockets extremely thoroughly more than 10 times and still have not found my keys. Therefore, it is safe to conclude that my keys are not in my coat pockets.

7. I was able to read the book “Where the Wild Things Are” in one weekend. Therefore, I should be able to read the novel “War and Peace” in a weekend as well.
8. There is a lot of social pressure to implement stricter censorship codes on prime time television these days. But there are actually a lot of good programs on the air at that time. Reality TV shows and the evening news are both on during prime time.
9. I have seen so many people driving hybrid vehicles lately. They must be really good cars.
10. At what point in the semester did you decide that Professor Vaidya was the best teacher you have ever had in your entire life?
11. Susan is pretty tall. So, her kids will grow up to be tall, too.
12. The last person to take a shower before the drain started to get backed up was Brian. He must be to blame for clogging it all up.
13. Lionel has looked around a little, but he still can’t seem to find where in the mall parking lot he parked his car. He must’ve not have parked in a different lot.
14. Marcus was the only witness to the crime. But Marcus has been diagnosed by his psychiatrist with schizophrenia, narcolepsy, neurosarcoidosis, and bell mania and was under the influence of crack, peyote, LSD, Oxycontin, and had a blood alcohol level of .35 at the time of the crime. Therefore, Marcus’ testimony should not count as evidence.
15. While you live in my house, you will believe in the values I believe in, otherwise I will kick you out on the street.
16. If a particular substance is extremely detrimental to the health of those who use it, then the legislature should make that substance illegal. Therefore, the legislature should make coarse ground black pepper illegal.
17. Some people believe that religion should not be taught in elementary schools. But this would amount to firing any teacher who decides to teach religion anyway. Apparently these people are just hostile towards teachers and want them to lose their jobs. We shouldn’t believe them.

18. I overheard some guy on the street saying something about how purified water can cause liver failure. Purified water must be really bad. I'll never drink that stuff again.
19. So far, the Mets have lost every single game this season. Therefore they're bound to win at least one of the next few games.
20. Some politicians say they are in favor of releasing people from prison who were only convicted of non-violent drug offenses. But if we did that, then there would probably be more politicians arguing for releasing people who have committed armed robbery, followed by a movement to release and rapists and murderers. Pretty soon our streets would be flooded with dangerous people. So, we shouldn't release non-violent drug offenders.
21. Last spring semester, it was sunny while I was taking my final and I ended up getting all A's. Therefore, if it's sunny while I'm taking this final, I should get an A also.
22. Jason's grades have been slipping down lately. So, he should just drop out of school.
23. All of the houses downtown are historic houses, which means they are all at least 75 years old. So it must be the case that all of the light fixtures inside are at least 75 years old as well.
24. It seems that all the rich and classy people at the party had nice bowties. Therefore, if I ever want to be rich and classy, I should get myself a bowtie.
25. Martin told Erin that she should quit smoking because it is expensive, it is bad for her health, it hinders her ability to exercise, it supports big tobacco corporations, it is unattractive to many people and it makes her hair and clothes smell bad. But Martin used to smoke himself. So, Erin shouldn't heed any of his admonitions.
26. Tim, his wife, and both of their mothers, fathers and grandparents on both sides of their families are all tall. Therefore, Tim and his wife's baby is probably going to be tall as well.

27. The people of this city should vote ‘no’ on the proposition on the ballot that would dam the river because damming that river would be a bad idea.
28. My grandpa told me that when he was a kid about 60 years ago, he used to go to a small ice cream shop called ‘Double Scoop’ every Saturday. Today is Saturday, so Double Scoop must be open today.
29. Jesse just read Steinbeck’s “Grapes of Wrath,” and told me that it was an incredible book that I just have to read. But Jesse is one of the dumbest, most illiterate people I know. He couldn’t tell a timeless brilliant piece of literature from a sports article in the monthly junior high school news publication. So, “Grapes of Wrath” is probably a terrible book and I’m not going to read it.
30. It’s illegal to punch people unless it’s in self defense. So, boxers and UFC fighters should be arrested.
31. A lot of the small businesses in the country have been struggling lately. That must be because of a growing inability of small business owners to make good business decisions.
32. I saw an advertisement that claimed that their diet program could give me a gorgeous body in just 4 hours. I sure wish I had an attractive body. This diet program sounds like it would work perfectly for me.
33. People shouldn’t be afraid of dying. After all, dying isn’t the end; we’ll all go to heaven or be reincarnated or something like that.
34. Either you graduate college or you become a bum for the rest of your life. It’s as simple as that.
35. On rainy days, Sally says she likes to stay at home all day and just listen to the rain fall in her bedroom. She must have a leaky roof.
36. Sheila always eats very rich desserts. She must have a lot of money.
37. The prison workers’ union is demanding more substantial pension packages for their workers. But if we provide them with the funding they are demanding, the next thing they’re going to ask for is greater health benefits, and then higher salaries, and

pretty soon the state is going to go bankrupt trying to pay for all of this. We should not give them the funding for their pension packages.

38. Many politicians today are considering policies that put more restrictions on the selling and purchasing of tobacco products. But these policies are basically policies that have absolutely no respect for autonomy and our right to choose what to put in our own body. And any policy that is so radical that it wants to completely throw away all of our individual freedom is a ridiculous policy.
39. Scientists still haven't found a cure for Alzheimer's disease and they have been searching for a long time. It must be an incurable disease.
40. The weather forecast says it should be hot 95°F in San Jose today. Both San Jose and San Francisco are in the bay area. In fact, the two cities are only a quick 40 miles apart. So, San Francisco should be pretty hot today as well.

CH 4: Criteria for Evaluating Arguments

Answer the following questions:

1. Explain the distinction between the form of an argument and the content of an argument. Given an example in your explanation.

2. Is validity a formal property of an argument or a content based property of an argument?

3. Explain why a sound argument cannot have a false conclusion. Your explanation can take the form of an argument involving the definitions of validity and soundness.

4. Give an example of a valid argument with an actually false premise and an actually true conclusion.

5. Explain why an argument can fail to be good even though it is valid and sound. Provide an example.

6. What are the two kinds of support that the inferential claim of an argument can make? Explain the difference between the two with respect to what the author of the argument intends to be the case.

7. State the definition of an invalid argument.

8. What are the two ways in which an argument can be rendered unsound?

9. Is soundness a formal property of an argument, a content based property of an argument, or both?

10. Can a valid argument have all actually true premises and an actually false conclusion?

Ch 5: The Language of Propositional Logic

Using the strict definition of a well-formed formula circle the strings that are well-formed:

(a) $\rightarrow ($P

(b) $\equiv ($P $\neg$

(c) (P $\rightarrow$ Q)

(d) (P $\wedge$ Q) $\rightarrow$ (R $\vee$ S)

(e) $\rightarrow\neg\equiv$P

(f) (P $\wedge$ Q ($\vee$ R

(g) (P $\rightarrow$ Q

(h) $\neg$P $\rightarrow$ (Q$\wedge$R)

(i) (P $\rightarrow$ Q) $\rightarrow$ R $\rightarrow$ S

(j) (P $\rightarrow$ Q $\rightarrow$ R)

Determine the main connective of each of the following formulas:

(1) ((P $\rightarrow$ Q) $\wedge$ R)

(2) $\neg$(P $\rightarrow$ Q)

(3) ($\neg$(P $\rightarrow$ Q) $\equiv$ (R $\vee$ S))

(4) ((P $\wedge$ Q) $\vee$ (R $\wedge$ S)) $\rightarrow$ T)

(5) $\neg$(((P $\vee$ S) $\rightarrow$ Q) $\equiv$ R)

(6) ((P $\rightarrow$ Q) $\rightarrow$ (R $\rightarrow$ S))

(7) ((P $\rightarrow$ Q) $\vee$ (T $\vee$ S))

(8) ((P $\wedge$ Q) $\wedge$ (S $\wedge$ T))

(9) ((R $\equiv$ S) $\rightarrow$ $\neg$T)

(10) (((P $\wedge$ Q) $\rightarrow$ T) $\vee$ S)

Explain the difference between capital letters that are elements of the language of PL, and lower case letters that are strictly speaking not part of the language of PL.

Write out the truth-functional definition for each of the following formulas:

(1) $(P \rightarrow Q)$

(2) $(P \vee Q)$

(3) $(P \wedge Q)$

(4) $\neg P$

(5) $(P \equiv Q)$

Why are the connectives and operators of propositional logic called truth-functional? Explain your answer via a mathematical analogy.

Translation Guide:

Statement Letter of PL: P, Q, R, S, T, U, V, X, Y, Z.
Logical Operators of PL: ≡, →, ¬, ∧, ∨
Rules of Syntax for PL:
Let *p* and *q* be meta-linguistic variables ranging over well-formed formulas (WFFs) of PL.

1. **Every Statement Letter of PL is a WFF**
2. **If *p* and *q* are WFF, then so is**
   (i) $\neg p$
   (ii) $(p \wedge q)$
   (iii) $(p \vee q)$
   (iv) $(p \rightarrow q)$
   (v) $(p \equiv q)$
3. **Nothing is a WFF unless it follows from (1) and (2).**
   Semantic of PL:

1. **$\neg p$ is true when *p* is false, and false when *p* is true.**
2. **$(p \wedge q)$ is true just in case both *p* and *q* are true, false otherwise.**
3. **$(p \vee q)$ is false just in case both *p* and *q* are false, true otherwise.**
4. **$(p \rightarrow q)$ is false just in case *p* is true and *q* is false, true otherwise.**
5. **$(p \equiv q)$ is true just in case either both *p* and *q* are true or both *p* and *q* are false.**

English	**Translation**
It is not the case that P.	$\neg$P
P and Q. P but Q. Although P, Q.	(P ∧ Q)
Either P or Q. P or Q.	(P ∨ Q)
If P, then Q.	(P → Q)
P if Q.	(Q → P)
P only if Q.	(P → Q)
P if and only if Q.	(P ≡ Q)
P just in case Q.	(P ≡ Q)

Neither P nor Q.	$\neg(P \vee Q)$
It is not the case that both P and Q.	$\neg(P \wedge Q)$
Both not P and not Q.	$(\neg P \wedge \neg Q)$
Only if P, Q.	$(Q \rightarrow P)$
P is necessary for Q.	$(Q \rightarrow P)$
P is sufficient for Q.	$(P \rightarrow Q)$
P is both necessary and sufficient for Q.	$(P \equiv Q)$
P unless Q	$(P \vee Q)$ or $(\neg P \rightarrow Q)$

CH 6: Translation Workbook

Translate the following English statements into PL using a scheme of abbreviation.

1. Either John is tall or Mary is short.

2. Neither John nor Mary passed their exam.

3. It is not the case that Mary went to school.

4. Both Jim and Bill went to practice.

5. It is not the case that either Bill or Mary joined the army.

6. Both Zidane and Beckham play for Real Madrid.

7. It is not the case that both Zidane and Ribery will play for France.

8. Neither France nor Germany will play for 2nd place.

9. Jim is not running this week.

10. Mary and Tom are going to the park.

11. Mary will not join the team and will not play the flute.

12. Regular exercise and having a balanced diet are good for one's health.

13. John will go to school, and either Mary will not go to the park or Jim will.

14. John, and either Mary or Frank will attend church today.

15. Either soccer fans are mellow or they are hyper excited.

16. Mary and Bethany will not try out for the team.

17. It is not true that Mary and Bethany will not try out for the club.

18. Either Bethany will not win or Mary will not win.

19. It is not the case that Mary will not win.

20. Either Mary won't win, or Jim and Bill will go on to the semifinals.

21. If Mary goes to church, Bill will join the choir.

22. John will go to school only if Sue goes to school.

23. Having a driver's license is a necessary condition for driving a car.

24. A sufficient condition for being a US citizen is being born in the US.

25. John will join the team, if Mark joins the team.

26. If John does not pass the test, then John will have to take it again.

27. If Mary isn't the daughter of Samantha, then it is not the case that Mary is the daughter of Bill.

28. Only if Mary wins the Election, will Bill run the following year.

29. Just in case Jim gets an A on every exam, will Jim get an A in the class.

30. John is responsible for the crime only if John is a free agent.

31. If John has a justified true belief that a square is a rectangle, then John knows that a square is a rectangle.

32. A necessary and sufficient condition for John being a member of the X club is being over 21.

33. Mary is going to school if and only if Mary wakes up.

34. John is going to win the election if and only if John wins 38 states.

35. France will win the world cup if and only if France beats Italy.

36. John will win the election only if Mary withdraws.

37. If France wins group B, then if Germany wins Group A, Germany will play France only if Italy has less points than France

38. Germany will play France only if Italy has fewer points than France.

39. If John goes to school, then Mary will stay home, only if Bill will ride his bike.

40. France advances, if Spain beats Germany.

Translate the following English statements into PL using the following scheme of abbreviation -- Z: Zidane will play for FC Barcelona. R: Ribery will play for France. P: Pauletta will play for Germany. T: Tevez will play for Arsenal.

41. Zidane will play for FC Barcelona; only if either Tevez will play for Arsenal or Pauletta will play for Germany.

42. Either Tevez will play for Arsenal or Ribery will play for France, if and only if Pauletta will play for Germany and Zidane will play for FC Barcelona.

43. Neither Ribery will play for France nor Zidane will play for FC Barcelona, if Tevez will play for Arsenal.

44. It is not the case that both Tevez will play for Arsenal and Pauletta will play for Germany, if either Zidane will play for FC Barcelona or Ribery will play for France.

45. Neither Pauletta will play for Germany, if Tevez will play for Arsenal; nor Zidane will play for FC Barcelona, if Ribery will play for France.

46. It is not the case that Pauletta will play for Germany, only if Tevez will play for Arsenal.

47. Zidane will not play for FC Barcelona, if and only if Pauletta will play for Germany.

48. Only if Pauletta plays for Germany, will Zidane not play for FC Barcelona.

49. If neither Zidane plays for FC Barcelona nor Ribery plays for France, then Pauletta will not play for Germany.

50. If Zidane will play for FC Barcelona, then either Pauletta will play for Germany or Ribery will not play for France.

Translate the following English statements into PL using the following scheme of abbreviation – P: Mary wins the primary. Q: Bill wins the secondary. R: Susan beats John. S: Lisa beats Frank. T: Tom wins the final.

51. It is not the case that Tom wins the final, if either Mary wins the primary or Susan beats John.

52. Either Susan beats John or Lisa beats Frank, only if both Mary wins the primary and Bill wins the secondary.

53. Neither Susan will beat John nor Bill win the secondary, if either Lisa beats Frank or Mary wins the primary.

54. It is not the case that Susan will beat John and it is not the case that Mary will win the primary.

55. It is not the case that Tom will win the final if and only if either Mary wins the primary or Susan beats John.

56. Either Susan beats John or if Mary wins the primary, Lisa will beat Frank.

57. It is not the case that Mary wins the primary, Bill wins the secondary, and Susan beats John.

58. If Lisa beats Frank, then Tom wins the final only if Mary wins the primary.

59. Tom will win the final unless Mary wins the primary and Susan beats John.

Truth-Table Guide

Truth-Tables:

The number of rows that a truth-table requires is determined by the formula 2^n, where n = the number of basic statement letters. If you have 3 basic statement letters, P, Q, R, then you need $2^n = 8$. The easiest way to construct a truth-table without missing a possible combination of truth values is to make the first column half true and half false, and the next column half true and half false, and so on down for each column. For

Logical Relation	**Truth-Table**
Contingent	Some T and Some F
Tautologus	All T
Contradictory	All F
Equivalent	Every row has the same truth-

example, if you have 3 columns, you have 8 rows. The first column gets 4 'T', followed by 4 'F'. The next column gets 2 'T' followed by 2 'F' followed by 2 'T' followed by 2 'F'. The last column gets 'T', 'F' alternating every row.

	value.
Satisfiable/Consistent	At least one row where every formula is true.
Unsatisfiable/Inconsistent	No row where every formula is true.
Valid	No row where premises are T, and conclusion is F.

Ch 7: Truth-Table Analysis

Using a complete truth-table determine for each of the following wffs whether the wff is tautologus, contingent, or contradictory.

1. $(P \rightarrow (Q \rightarrow P)$
2. $(\neg P \rightarrow P)$
3. $(P \rightarrow (P \vee (Q \vee R)$
4. $\neg \neg P$
5. $\neg (P \wedge Q)$

Using a complete truth-table determine for each of the following sets of wffs whether the set contains equivalent wffs, satisfiable

wffs, or unsatisfiable wffs. Note: a set of wffs may be both equivalent and satisfiable or equivalent and unsatisfiable.

6. {(P → Q), (¬P ∨ Q)}
7. {¬(¬P ∧ ¬Q), (P ∨ Q)}
8. {(P ≡ Q), ((P ∨ Q) ∧ ¬(P ∧ Q))}
9. {(P→ Q), (Q→P)}
10. {¬(P → Q), (P ∧ ¬Q)}

Using a complete truth-table determine whether the following arguments are valid.

11. (P ∨ Q), ¬P / Q
12. P/ (P ∨ Q)
13. (P → Q), ¬P / ¬Q
14. (P → Q, ¬Q / ¬P
15. (P → (Q → R), (P → Q) / (Q → R)

Explain how the concept of validity is captured by a truth-table test for validity. (Hint: think about what rows and columns represent relative to the definition of an invalid argument.).

CH 8: Indirect-Table Analysis

Using the indirect-table method determine whether the following sets of propositions are satisfiable.

1. {(P → Q), (¬P ∨ Q)}
2. {¬(¬P ∧ ¬Q), (P ∨ Q)}
3. {(P ∧ Q), (P ∨ Q)}
4. {(P ≡ Q), (¬P ∧ ¬Q)}
5. {(P → Q), (P ∧ ¬Q)}

Using the indirect-table method determine whether the following arguments are valid.

6. $(P \rightarrow Q), \neg P \, / \, \neg Q$
7. $(P \equiv Q), \neg Q \, / \, \neg P$
8. $P \, / \, (P \wedge Q)$
9. $(P \rightarrow Q), (Q \rightarrow R) \, / \, (P \rightarrow R)$
10. $(P \vee Q), (Q \vee R) \, / \, (P \vee R)$

Explain how the indirect-table method works in the case of proving satisfiability, and validity. (Hint: think about the relation between an indirect-table and a complete table.).

Truth-Tree Rule Guide

A tree-test is a consistency test for a given input stack. The stack is the input, and by applying the rules one breaks down all complex formulas into either affirmations or negations of atomic statement letters.

Closed Path Rule: if an formula and its negation appear on the same path, then there is a contradiction on the path, and thus the path is closed. If there is no contradiction on a given path, the path remains open.

Consistent: A stack is consistent when there is at least one open path.

Inconsistent: A stack is inconsistent when there are no open paths.

Validity: to test whether an argument is valid negate the conclusion of the argument, and check to see whether the stack containing the original premises and the negation of the conclusion is consistent. If the negation of the conclusion and the original set of premises leads to at least one open path, then the original argument is invalid. However, if the original premises, and the negation of the conclusion lead to all closed paths, then the original argument is valid.

Tautology: to test whether a formula is a tautology, negate the formula and check to see if the negation is a contradiction. If the negation of the formula is a contradiction, then the original formula is a tautology. However, if there is at least one path that is open, then the negation of the formula is consistent, and so the original formula is not a tautology.

Logical Equivalence: to test whether to formulas are logically equivalent create a biconditional between the two formulas and test to see whether the biconditional is a tautology. In particular, negate the biconditional, check to see whether every path closes. If every path closes, then the original formula is a tautology, and the two formulas in particular are logically equivalent.

Rules for Truth-Tree Construction

1. Any time a formula occurs in a tree and the negation (if it is a negation, its affirmation) of the formula at some line below it on a connected branch one places an X underneath the formula, signifying that the branch is closed.

$$\begin{array}{c} \bigcirc \\ \underline{\neg\bigcirc} \\ \text{X} \end{array}$$

2. A formula with two negations applying directly to it can be replaced on a line by the formula itself.

¬ ¬ ○
○

3. A negated conditional is broken down into a single trunk with the affirmation of the antecedent followed by the negation of the consequent.

✓¬ (○ → △)
○
¬ △

4. A conditional is broken down into two separate trunks, one trunk with the negation of the antecedent, and one with the affirmation of the consequent.

✓(○ → △)
¬○   △

5. A conjunction is broken down into a single trunk with the affirmation of both conjuncts stacked.

✓(○ ∧ △)
○
△

6. A disjunction is broken down into two trunks, one trunk for each disjunct.

✓(○ ∨ △)
○   △

7. A biconditional is broken down into two trunks, one trunk has both the affirmation of the antecedent and the consequent, and the second trunk has the negation of both the antecedent and consequent.

✓(○ ≡ △)
○   ¬△
△   ¬○

8. A negated conjunction is broken down into two trunks; each trunk has a negation of one of the conjuncts.

✓¬(○ ∧ △)
¬○ ¬△

9. A negated disjunction is broken down into a single trunk with each disjunct negated.

✓¬(○ ∨ △)
¬○
¬△

10. A negated biconditional is broken down into two trunks, one trunk contains the negation of the antecedent, and the affirmation of the consequent, the other trunk contains the affirmation of the antecedent, and the negation of the consequent.

✓¬(○ ≡ △)
¬○ ○
△ ¬△

Ch 9: Truth-Tree Analysis

Using the truth-tree method, determine whether the following sets of formulas are satisfiable / consistent.

1. {(P → Q), ¬(¬P ∨ Q)}
2. {(P ≡ Q), ((P ∨ Q) ∧ ¬(P ∧ Q))}
3. {(P ∨ (Q ∨ R), (¬Q ∧ ¬R)}
4. {(P → (Q → R), P, ¬R}
5. {(P ∧ Q), (Q ∨ R)}

Using the truth-tree method, determine whether the following arguments are valid.

1. $(P \rightarrow Q), \neg Q$ / P
2. $(P \rightarrow Q), (R \rightarrow S), (\neg S \vee \neg Q)$ / $(\neg R \vee \neg P)$

(MP)	(MT)
$(p \rightarrow q), p$ / q	$(p \rightarrow q), \neg q$ / $\neg p$

3. $(P \vee (Q \vee R), (\neg Q \wedge \neg R)$ / P
4. $(P \rightarrow (Q \rightarrow R), P, \neg R$ / $\neg Q$
5. $(P \wedge Q)$ / $(Q \vee R)$
6. $\neg(P \rightarrow Q)$ / $(\neg Q \vee R)$
7. $(P \vee \neg Q), (\neg P \vee R)$ / $(\neg R \rightarrow \neg P)$
8. $(P \rightarrow Q), (Q \rightarrow P)$ / $(P \equiv Q)$
9. $(P \wedge Q), (Q \wedge \neg R)$ / $(R \rightarrow \neg P)$
10. $(P \vee \neg P) \rightarrow (Q \rightarrow R)$

Explain how the truth-tree method tests for validity.

Natural Deduction Rule Guide

Rules of Inference:

Rules of Replacement:

(HS) $(p \rightarrow q), (q \rightarrow r) / (p \rightarrow r)$	(DS) $(p \vee q), \neg p / q$
(Add) $p / (p \vee q)$	(Con) $p, q / (p \wedge q)$
(Simp) $(p \wedge q) / p$	(CD) $(p \vee q), (p \rightarrow r), (q \rightarrow s) / (r \vee s)$

(DN) $\neg\neg p :: p$	(Imp) $(\neg p \vee q) :: (p \rightarrow q)$
(Cont) $(p \rightarrow q) :: (\neg q \rightarrow \neg p)$	(DM) $\neg(p \wedge q) :: (\neg p \vee \neg q)$ $\neg(p \vee q) :: (\neg p \wedge \neg q)$
(Exp) $(p \rightarrow (q \rightarrow r)) :: ((p \wedge q) \rightarrow r)$	(Equiv) $(p \equiv q) :: ((p \rightarrow q) \wedge (q \rightarrow p))$
(Cont) $(p \rightarrow q) :: (\neg q \rightarrow \neg p)$	(Taut) $p :: (p \vee p)$
(Comm) $(p \vee q) :: (q \vee p)$ $(p \wedge q) :: (q \wedge p)$	(Asc) $(p \vee (q \vee r) :: (p \vee q) \vee r)$ $(p \wedge (q \wedge r) :: (p \wedge q) \wedge r)$
(Dist) $(p \wedge (q \vee r) :: (p \wedge q) \vee (p \wedge r)$ $(p \vee (q \wedge r) :: (p \vee q) \wedge (p \vee r)$	

Conditional Proof (CP): Assume the antecedent p of $(p \rightarrow q)$, derive q using only premises and valid rules of inference enter $(p \rightarrow q)$ on the line after q, with the justification (CP) and the lines used.

Indirect Proof (IP): Assume the negation of the conclusion. So, if the conclusion is p, assume $\neg p$. Use $\neg p$, the premises, and any valid rules of inference to derive a contradiction $\perp$ of the form $(r \wedge \neg r)$. On the line after the contradiction is formed enter p, with justification (IP)

Ch 10: Rules of Inference

Using the rules of inference prove the following:

1. $P \rightarrow (Q \rightarrow R), \neg(Q \rightarrow R), (\neg P \vee S) \rightarrow T / T$
2. $(P \rightarrow \neg Q), (\neg Q \rightarrow R), T, (P \rightarrow R) \wedge T) \rightarrow S / S$
3. $(P \vee Q), \neg P, (Q \rightarrow S), (\neg R \rightarrow \neg T), T / (R \wedge S)$

4. $P, Q, (\neg(P \wedge Q) \vee R), R \rightarrow (S \rightarrow T), (S \rightarrow T) \rightarrow W) \;/\; W$
5. $(P \wedge Q), P \rightarrow R, Q \rightarrow S, (R \wedge S) \rightarrow (W \vee T) \;/\; W \vee T$
6. $(P \rightarrow Q), (\neg Q \wedge R), (\neg P \rightarrow S), (R \rightarrow V) \;/\; (S \wedge V)$
7. $(P \vee R), (\neg P \wedge S), (R \wedge S) \rightarrow \neg V), (V \vee Z) \;/\; Z$
8. $(P \wedge \neg S), P \rightarrow (S \vee V), (V \rightarrow Z) \;/\; Z$
9. $P, (P \vee Q) \rightarrow \neg Z), (Z \vee V), (V \rightarrow R) \;/\; R$
10. $(P \rightarrow Z), (\neg Z \wedge V), (\neg P \wedge V) \rightarrow R) \;/\; R$

Ch 11: Rules of Replacement

Using the rules of replacement and the rules of inference prove the following.

1. $(\neg R \vee S), (P \vee Q) \rightarrow (R \wedge \neg S) \;/\; \neg P$
2. $(P \wedge (Q \rightarrow R), \neg(R \wedge P) \;/\; \neg Q$

3. $(H \rightarrow K), (C \equiv D), (\neg C \rightarrow \neg K) \ / \ (H \rightarrow D)$
4. $(\neg P \rightarrow (Q \wedge R), \neg R \ / \ P$
5. $(\neg R \vee \neg S), (P \rightarrow (R \wedge S) \ / \ \neg P$
6. $\neg P, (P \vee Q) \equiv R), \neg Q \ / \ \neg (R \wedge S)$
7. $(P \rightarrow Q), (Q \wedge R) \rightarrow S) \ / \ (Q \rightarrow (R \rightarrow S)$
8. $(P \vee (Q \wedge R), \neg R \ / \ P$

CH 12: Proof Rules

Using conditional proof for conclusions with conditionals and indirect proof for conclusions that are not conditionals prove:

1. $(R \vee S) \rightarrow T), (P \vee Q) \rightarrow T), (R \vee P) \ / \ T$
2. $(P \rightarrow (Q \rightarrow P)) \rightarrow S) \ / \ S$
3. $(S \vee T) \rightarrow \neg S \ / \ \neg S$
4. $(P \rightarrow P) \rightarrow R), (R \vee S) \rightarrow Q) \ / \ Q$

5. $(P \rightarrow (Q \vee R), (P \rightarrow R) \rightarrow (S \wedge T), (Q \rightarrow R) \;/\; T$
6. $(Q \rightarrow (R \rightarrow S), (Q \rightarrow (T \rightarrow \neg U), (U \rightarrow (R \vee T) \;/\; (Q \rightarrow (U \rightarrow S)$
7. $(P \rightarrow Q), (P \rightarrow R) \;/\; P \rightarrow (Q \wedge R)$
8. $\neg(U \wedge W) \rightarrow X), (U \rightarrow \neg U) \;/\; \neg(U \vee \neg X)$

Answer Key

Chapter 1:

1. Explanation
2. Illustration
3. Argument
4. Report
5. Argument

6. Illustration
7. Argument
8. Report
9. Explanation
10. Argument

Chapter 2:

1. (1)If John runs for mayor, then so will Mary. (2)John will run for mayor. (3)Therefore, so will Mary.

$$\frac{(1) + (2)}{(3)}$$

2. (1)Smoking is bad for your health. (2) It makes lung disease more likely. (3)It also ruins your teeth.

$$\frac{(2)\quad(3)}{(1)}$$

3. (1)Exercise helps prevent high blood pressure, (2)it reduces the risk of various kinds of cancer, and (3)it helps control body weight. Apparently(4) exercise is good for your health.

$$\frac{(1)\quad(2)\quad(3)}{(4)}$$

4. (1)If John asks Mary to the prom, then either Justin or Carlos will ask Aurora to the prom. (2)John will ask Mary to the prom. So, (3)either Justin or Carlos will ask Aurora to the prom. But (4)Justin will not ask Aurora to the prom. So, (5)Carlos will ask Aurora to the prom.

$$\frac{\dfrac{(1) + (2)}{(3)} \quad + \quad (4)}{(5)}$$

5. (1)Osama bin Laden is living either in Pakistan or Iran. (2)He cannot be living in Iran because (3)an anonymous informant said he saw bin Laden leaving his hide-out in Iran and heading for the Pakistani border. So, (4)bin Laden must be living in Pakistan.

$$\begin{array}{l}\underline{(3)}\\ \underline{(2) + (1)}\\ \quad\underline{(4)}\end{array}$$

6. (1)All politicians are greedy. (2)Mike Pankrast is a politician. So, (3)Mike Pankrast is greedy. (4)All greedy people are bad people. So, (5)Mike Pankrast is a bad person.

$$\begin{array}{l}\underline{(1) + (2)}\\ \quad\underline{(3)\quad + \quad(4)}\end{array}$$

(5)

7. (1)The possession of firearms should be illegal. (2)Allowing the ownership of firearms has always and will always raise the incidents of gun-related deaths. (3)The reason why this is the case is because when firearms have a widespread legal distribution, there is a greater chance that people will irresponsibly use them in situation in which they feel threatened. (4)It's just human nature to behave this way.

$$\begin{array}{c}\underline{(4)}\\ \underline{(3)}\\ \underline{(2)}\\ (1)\end{array}$$

8. (1)If we elect candidate Julie Olsen as mayor our crime rate will go up, because (2)she plans to cut law-enforcement funding and because (3)she plans to pass a city ordinance that would make it illegal to loiter downtown. So, (4)we shouldn't elect candidate Julie Brown as mayor, because (5)we don't want our crime rate to go up.

$$\begin{array}{c}\underline{(2) + (3)}\\ \underline{(1) \quad + \quad (5)}\\ (4)\end{array}$$

9. (1)If there is to be a more globally peaceful state of being for humanity, there must be less violence. (2)In order for there to be less violence, there must less starvation and malnutrition due to economic oppression. (3)In order for there to be less starvation and malnutrition due to economic oppression, the people, governments and corporations in more affluent countries must be willing to abdicate some of the privileges they enjoy due to an inequitable distribution of financial power in the international economy. (4)But the people, governments and corporations in more affluent countries are not willing to abdicate any of the privileges they enjoy due to an inequitable distribution of financial power in the international economy. (5)Therefore, there will not be a more globally peaceful state of being for humanity.

$$\frac{(1) + (2) + (3) + (4)}{(5)}$$

10. (1)The murderer of Dr. Thornburg must be Ms. Davenport. (2)The only three possible suspects were Professor Gunther, Mr. Ivanovich, and Ms. Davenport, and (3)it could not possibly have been Professor Gunther and (4)it could not possibly been Mr. Ivonovich. (5)Professor Gunther was infirm in the hospital during the murder. Furthermore, (6)she had no motive. And (7)Mr. Ivanovich was in Hawaii during the time of the murder. Besides, (8)Dr. Thornburg was shot with a .45 caliber hand gun, and (9)that same hand gun was found in Ms. Davenport's car with her fingerprints on it.

$$\cfrac{\cfrac{(5)\ \ (6)}{(3)} + \cfrac{(7)}{(4)} + (2) \qquad (8) + (9)}{(1)}$$

Chapter 3: Fallacious Reasoning in Argumentation

1. Composition
2. Ad hominem circumstantial
3. Begging the question (arguing in a circle)
4. Appeal to pity
5. Hasty generalization

6. No fallacy
7. Weak analogy
8. Red herring
9. Appeal to the people (bandwagon)
10. Complex question
11. Genetic fallacy
12. False cause (post hoc ergo propter hoc)
13. Appeal to ignorance
14. No fallacy
15. Appeal to force
16. Begging the question (leaving out key premise)
17. Straw man
18. Appeal to unqualified authority
19. Gambler's fallacy
20. Slippery slope
21. False cause (non causa pro causa)
22. Missing the point
23. Division
24. Appeal to the people (snobbery)
25. Ad hominem (tu quoque)
26. No fallacy
27. Begging the question (restating the conclusion as a premise)
28. Suppressed evidence
29. Ad hominem (abusive)
30. Accident
31. False cause (oversimplified cause)
32. Appeal to the people (vanity)
33. Begging the question (missing controversial premise)
34. False dichotomy / Complex question
35. Amphibole
36. Equivocation
37. Slippery slope
38. Straw man
39. Appeal to ignorance

40. Weak analogy

Chapter 4: Criteria for Evaluating Arguments

1. The content of an argument is what, specifically, the argument is about, whereas the form of an argument refers to only the structure of the argument.

   Example:

a) Either Jed or John will win the race. b) Either Josh or Jim will drive.
Jed will not win the race. Josh will not drive.
∴ Therefore, John will win the race. ∴ Therefore, Jim will drive.

Arguments (a) and (b) have different content. The content of (a) is about Jed and John winning a race or not, and the content of (b) is about Jim and Josh driving or not. However, both argument (a) and (b) have the same form:

Either P or Q
Not P
∴ Therefore Q

2. Validity is a formal property of an argument.
3. Since a valid argument is one in which it is impossible for the premises to be true and the conclusion false, and since a sound argument is a valid argument that, in addition to being valid has true premises, it follows that a sound argument must (according to the definitions of validity and soundness) have a true conclusion.
4. If San Jose is in California, then Mississippi is in Beijing.
   If Mississippi is in Beijing, then San Jose is in the United States.
   ∴ If San Jose is in California, then San Jose is in the United States.
5. An argument that is valid and sound can still fail to be good in the case that it utilizes circular reasoning, by restating the conclusion as a premise. There are instances of sound arguments that do this. Consider the following example:

   If I have 100 pennies, then I have 1 dollar.
   ∴ If I have 100 pennies, then I have 1 dollar.
6. An invalid argument is one in which it is possible for the premises to be true and the conclusion false.
7. An argument can be rendered unsound if either:
   1) any of its premises are false

2) it has an invalid form

8. Both
9. No

Chapter 5: The Language of Propositional Logic

Syntax Exercises

a. P
b. P
c. (P → Q)
d. P, Q, R, S, (P ∧ Q), (R ∨ S)
e. P
f. P, Q, R

g. P, Q
h. P, ¬P, Q, R, $(Q \wedge R)$
i. P, Q, R, S, $(P \rightarrow Q)$
j. P, Q, R

(1) $((P \rightarrow Q) \wedge R)$
(2) $\neg(P \rightarrow Q)$
(3) $(\neg(P \rightarrow Q) \equiv (R \vee S))$
(4) $((P \wedge Q) \vee (R \wedge S)) \rightarrow T)$
(5) $\neg(((P \vee S) \rightarrow Q) \equiv R)$
(6) $((P \rightarrow Q) \rightarrow (R \rightarrow S))$
(7) $((P \rightarrow Q) \vee (T \vee S))$
(8) $((P \wedge Q) \wedge (S \wedge T))$
(9) $((R \equiv S) \rightarrow \neg T)$
(10) $(((P \wedge Q) \rightarrow T) \vee S)$

Q: Explain the difference between capital letters that are elements of the language of PL, and lower case letters that are strictly speaking not part of the language of PL.
A: The capital letters in PL are used to name specific statements in natural language. Lower case letters, on the other hand, do not name specific statements. Rather, they are used as variables to represent statements in general.

Semantics Exercises

1. $(P \rightarrow Q)$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

(P →Q) is true when both P and Q are true.
(P→Q) is false when P is true and Q is false.
(P →Q) is true when P is false and Q is true.
(P →Q) is true when both P and Q are false.

2. (P ∨ Q)

P	Q	P ∨ Q
T	T	T
T	F	T
F	T	T
F	F	F

(P ∨ Q) is true when both P and Q are true.
(P ∨ Q) is true when P is true and Q is false.
(P ∨ Q) is true when P is false and Q is true.
(P ∨ Q) is false when both P and Q are false.

3. (P ∧ Q)

P	Q	P ∧ Q
T	T	T
T	F	F
F	T	F
F	F	F

(P ∧ Q) is true when both P and Q are true.
(P ∧ Q) is false when P is true and Q is false.
(P ∧ Q) is false when P is false and Q is true.
(P ∧ Q) is false when both P and Q are false.

4. ¬P

P	¬P
T	F
F	T

(¬P) is false when P is true.
(¬P) is true when P is false.

5. P ≡ Q

P	Q	P ≡ Q
T	T	T
T	F	F
F	T	F
F	F	T

(P ≡ Q) is true when both P and Q are true.
(P ≡ Q) is false when P is true and Q is false.
(P ≡ Q) is false when P is false and Q is true.
(P ≡ Q) is true when both P and Q are false.

Q: Why are the connectives and operators of propositional logic called truth-functional? Explain your answer via a mathematical analogy.
A: They are called truth-functional operators because their semantical role is given by the way they map truth-values to truth-values. Each operator of PL determines what the truth-value of a compound statement involving the operator is depending on what the truth-value is of the formulas making up the compound. A truth functional operator is just like an algebraic function, such as x^2. The function x^2 takes numbers as inputs and yields numbers as output. Put in 2 and you get 4, put in 3 and you get 9. The function x^2 is different from the function x^3 because the range of each function is different. When you put in 2 to x^2 you get 4, but when you put 2 into x^3 you get 8. Truth-functions in propositional logic are similar. The only difference is that the inputs are either truth or falsity, and the output is either truth or falsity.

Chapter 6: Translation

1. J = John is tall.
   M = Mary is tall.

   J ∨ ¬M

2. J = John passed his exam.
   M = Mary passed her exam.

   $\neg(J \lor M)$

3. M = Mary went to school.

   $\neg M$

4. J = Jim went to practice.
   B = Bill went to practice.

   $J \land B$

5. B = Bill joined the army.
   M = Mary joined the army.

   $\neg(B \lor M)$

6. Z = Zidane plays for Real Madrid.
   B = Beckham plays for Real Madrid.

   $Z \land B$

7. Z = Zidane will play for France.
   R = Ribery will play for France.

   $\neg(Z \land R)$

8. F = France will play for 2nd place.
   G = Germany will play for 2nd place.

   $\neg(F \lor G)$

9. J = Jim is running this week.

   $\neg J$

10. M = Mary is going to the park.
T = Tom is going to the park.

$M \wedge T$

11. J = Mary will join the team.
P = Mary will play the flute.

$\neg J \wedge \neg P$

12. R = Regular exercise is good for one's health.
B = Having a balanced diet is good for one's health.

$R \wedge B$

13. J = John will go to school.
M = Mary will go to the park.
K = Jim will go to the park.

$J \wedge (\neg M \vee K)$

14. J = John will attend church today.
M = Mary will attend church today.
F = Frank will attend church today.

$J \wedge (M \vee F)$

15. M = Soccer fans are mellow.
H = Soccer fans are hyper excited.

$M \vee H$

16. M = Mary will try out for the team.
B = Bethany will try out for the team.

$\neg M \wedge \neg B$

17. M = Mary will try out for the club.
B = Bethany will try out for the club.

$\neg(\neg M \wedge \neg B)$

18. B = Bethany will win.
M = Mary will win.

$\neg B \vee \neg M$

19. M = Mary will win.

$\neg\neg M$

20. M = Mary will win.
J = Jim will go on to the semifinals.
B = Bill will go on to the semifinals.

$\neg M \vee (J \wedge B)$

21. M = Mary goes to church.
B = Bill will join the choir.

$M \rightarrow B$

22. J = John will go to school.
S = Sue goes to school.

$J \rightarrow S$

23. H = having a driver's license
D = driving a car

$D \rightarrow H$

24. U = being a US citizen
B = being born in the US

$B \rightarrow U$

25. J = John will join the team.
M = Mark joins the team.

$M \rightarrow J$

26. P = John passes the test.
T = John will have to take the test again.

$\neg P \rightarrow T$

27. S = Mary is the daughter of Samantha.
B = Mary is the daughter of Bill.

$\neg S \rightarrow \neg B$

28. M = Mary wins the Election
B = Bill runs the following year.

$B \rightarrow M$

29. E = Jim gets an A on every exam.
C = Jim gets an A in the class.

$E \equiv C$

30. R = John is responsible for the crime.
F = John is a free agent.

$R \rightarrow F$

31. J = John has a justified true belief that a square is a rectangle.
K = John knows that a square is a rectangle.

$J \rightarrow K$

32. M = John is a member of the X club.
J = John is over 21.

$M \equiv J$

33. G = Mary is going to school.
W = Mary wakes up.

$G \equiv W$

34. E = John is going to win the election.
S = John wins 38 states.

$E \equiv S$

35. W = France will win the world cup.
B = France beats Italy.

$W \equiv F$

36. J = John will win the election.
M = Mary withdraws.

$J \rightarrow M$

37. B = France wins Group B.
A = Germany wins Group A.
G = Germany plays France.
I = Italy has less points than France.

$B \rightarrow (A \rightarrow (G \rightarrow I))$

38. G = Germany will play France.
I= Italy has fewer points than France.

$G \rightarrow I$

39. J = John goes to school.
M = Mary will stay home.
B = Bill will ride his bike.

$(J \rightarrow M) \rightarrow B$

40. F = France advances.
S = Spain beats Germany.

$S \rightarrow F$

41. $Z \rightarrow (T \vee P)$
42. $(T \vee R) \equiv (P \wedge Z)$
43. $T \rightarrow \neg(R \vee Z)$
44. $(Z \vee R) \rightarrow \neg(T \wedge P)$
45. $\neg\{(T \rightarrow P) \vee (R \rightarrow Z)\}$
46. $\neg P \rightarrow T$
47. $\neg Z \equiv P$
48. $\neg Z \rightarrow P$
49. $\neg(Z \vee R) \rightarrow \neg P$
50. $Z \rightarrow (P \vee \neg R)$
51. $(P \vee R) \rightarrow \neg T$
52. $(R \vee S) \rightarrow (P \wedge Q)$
53. $(S \vee P) \rightarrow \neg(R \vee Q)$
54. $\neg R \wedge \neg P$
55. $\neg\{T \equiv (P \vee R)\}$
56. $(R \vee P) \rightarrow S$
57. $\neg\{P \wedge (Q \wedge R)\}$
58. $S \rightarrow (T \rightarrow P)$
59. $T \vee (P \wedge R)$

Chapter 7: Complete Truth Tables

P	Q	P → (Q →P)
T	T	**T** T
T	F	**T** T
F	T	**T** F

F	F	**T** T	1.

Tautologous

2.

P	¬P → P
T	F **T**
F	T **F**

Contingent

3.

P	Q	R	P → [P ∨ (Q ∨ R)]
T	T	T	**T** T T
T	T	F	**T** T T
T	F	T	**T** T T
T	F	F	**T** T F
F	T	T	**T** T T
F	T	F	**T** T T
F	F	T	**T** T T
F	F	F	**T** F F

Tautologous

4.

P	¬¬P
T	T
F	F

Contingent

5.

P	Q	¬ (P ∧ Q)
T	T	**F** T
T	F	**T** F
F	T	**T** F
F	F	**T** F

Contingent

6.

P	Q	P → Q	¬ P ∨ Q
T	T	**T**	F **T**
T	F	**F**	F **F**
F	T	**T**	T **T**
F	F	**T**	T **T**

Equivalent, Satisfiable

7.

P	Q	¬ (¬ P ∧ ¬ Q)	P ∨ Q
T	T	**T** F F F	**T**
T	F	**T** F F T	**T**
F	T	**T** T F F	**T**
F	F	**F** T T T	**F**

Equivalent, Satisfiable

8.

P	Q	P ≡ Q	(P ∨ Q) ∧ ¬ (P ∧ Q)
T	T	**T**	T **F** F T
T	F	**F**	T **T** T F
F	T	**F**	T **T** T F
F	F	**T**	F **F** T F

Not Equivalent, Unsatisfiable

9.

P	Q	$P \rightarrow Q$	$Q \rightarrow P$
T	T	**T**	**T**
T	F	**F**	**T**
F	T	**T**	**F**
F	F	**T**	**T**

Not Equivalent, Satisfiable

10.

P	Q	$\neg(P \rightarrow Q)$	$P \wedge \neg Q$
T	T	**F** T	**F** F
T	F	**T** F	**T** T
F	T	**F** T	**F** F
F	F	**F** T	**F** T

Equivalent, Satisfiable

11. P1 P2 C

P	Q	$P \vee Q$	$\neg P$	Q
T	T	**T**	**F**	**T**
T	F	**T**	**F**	**F**
F	T	**T**	**T**	**T**
F	F	**F**	**T**	**F**

Valid

12. P1 C

P	Q	P	$P \vee Q$
T	T	**T**	**T**
T	F	**T**	**T**
F	T	**F**	**T**
F	F	**F**	**F**

Valid

13. P1 P2 C

P	Q	P → Q	¬ P	¬ Q
T	T	**T**	**F**	**F**
T	F	**F**	**F**	**T**
F	T	**T**	**T**	**F**
F	F	**T**	**T**	**T**

Invalid

14. P1 P2 C

P	Q	P → Q	¬ Q	¬ P
T	T	**T**	**F**	**F**
T	F	**F**	**T**	**F**
F	T	**T**	**F**	**T**
F	F	**T**	**T**	**T**

Valid

15. P1 P2 C

P	Q	R	P → (Q →R)	P→ R	Q → R
T	T	T	**T** T	**T**	**T**
T	T	F	**F** F	**F**	**F**
T	F	T	**T** T	**T**	**T**
T	F	F	**T** T	**F**	**T**
F	T	T	**T** T	**T**	**T**
F	T	F	**T** F	**T**	**F**
F	F	T	**T** T	**T**	**T**
F	F	F	**T** T	**T**	**T**

Invalid

Explain how the concept of validity is captured by a truth-table test for validity.

A: Given that an argument is valid only when it is impossible for the premises to be true and the conclusion to be false, and given that truth tables show every possible combination of truth values of propositions, truth tables therefore are able to show if there is a possible case in which the premises are true and the conclusion false.

Chapter 8: Indirect-Table Analysis

1.

P → Q	¬ P ∨ Q
F T F	TF T F
F T T	TF T T
T T T	FT T T

Satisfiable

2.

¬ (¬ P ∧ ¬ Q)	P ∨ Q
T FT F FT	T T T
T FT F TF	T T F
T TF F FT	F T T

Satisfiable

3.

P ∧ Q	P ∨ Q
T T T	T T T

Satisfiable

4.

P ≡ Q	¬ P ∧ ¬ Q
F T F	TF T TF

Satisfiable

5.

P → Q	P ∧ ¬ Q
T T F	T T TF

Unsatisfiable

6. P1 P2 C

P → Q	¬ P	¬ Q
F T T	TF	FT

Invalid

7. P1 P2 C

P ≡ Q	¬ Q	¬ P
T T T	TT	FT

Valid

8. P1 C

P	P ∧ Q
T	T F F
F	F F T
F	F F F

Invalid

9. P1 P2 C

P → Q	Q → R	P → R
T T T	T T F	T F F

Valid

10. P1 P2 C

P ∨ Q	Q ∨ R	P ∨ R
T T T	T T F	T F F
F T T	T T T	F F T
F T T	T T F	F F F

Invalid

Q: Explain how the indirect-table method works in the case of proving satisfiability, and validity. (Hint: think about the relation between an indirect-table and a complete table.).

A: A set of statements is satisfiable when each of the statements are possibly true together. An indirect truth table, unlike a complete truth table which shows every possible truth combination, shows only the rows which are true under the main operator. If no contradiction arises when assigning a true truth value to each of the statements, then they are satisfiable.

An argument is valid when it is impossible for the premises to be true and the conclusion false. An indirect truth table, unlike a complete truth table which shows every possible truth combination, shows only the rows in which the premises are true under the main operator and the conclusion is true under the main operator. If no contradiction arises when assigning these truth values, then the argument is invalid.

Chapter 9: Truth Tree Analysis for Consistency

1.

$$
\begin{array}{ccc}
 & P \rightarrow Q & \\
 & \neg(\neg P \lor Q) & \\
 & \neg\neg P & \\
 & \neg Q & \\
\neg P & & Q \\
\text{X} & & \text{X} \quad \text{Inconsistent}
\end{array}
$$

2.

$$P \equiv Q$$

$(P \vee Q) \wedge \neg(P \wedge Q)$
$P \vee Q$
$\neg P \vee \neg Q$
P $\quad\quad$ $\neg Q$
Q $\quad\quad$ $\neg P$
$\neg P$ $\quad$ $\neg Q$ $\quad$ $\neg P$ $\quad$ Q
X $\quad$ X $\quad$ P $\quad$ Q $\quad$ P $\quad$ Q
X $\quad$ X $\quad$ X $\quad$ X $\quad$ Inconsistent

3. $P \vee (Q \vee R)$
$\neg Q \wedge \neg R$
$\neg Q$
$\neg R$
P $\quad\quad$ $Q \vee R$
Q $\quad$ R
X $\quad$ X $\quad$ Consistent

4. $P \rightarrow (Q \rightarrow R)$
P
$\neg R$
$\neg P$ $\quad\quad$ $Q \rightarrow R$
X $\quad$ $\neg Q$ $\quad$ R
X $\quad$ Consistent

5. $P \wedge Q$
$Q \vee R$
P
Q
Q $\quad$ R $\quad$ Consistent

Truth Tree Analysis for Validity

1. $P \rightarrow Q$

$\neg Q$

$\neg P$

$\neg P$ $\quad\quad$ Q

$\quad\quad\quad$ X $\quad\quad$ Invalid

2. $P \rightarrow Q$

$R \rightarrow S$

$\neg S \vee \neg Q$

$\neg(\neg R \vee \neg P)$

$\neg\neg R$

$\neg\neg P$

$\neg P$ $\quad\quad\quad\quad$ Q

X $\quad\quad$ $\neg R$ $\quad\quad$ S

$\quad\quad$ X $\quad$ $\neg S$ $\quad$ $\neg Q$

$\quad\quad\quad\quad$ X $\quad$ X $\quad\quad$ Valid

3. $P \vee (Q \vee R)$

$\neg Q \wedge \neg R$

$\neg P$

$\neg Q$

$\neg R$

P $\quad\quad\quad\quad$ $Q \vee R$

X $\quad\quad$ Q $\quad\quad$ R

$\quad\quad$ X $\quad\quad$ X $\quad\quad$ Valid

4. $P \rightarrow (Q \rightarrow R)$

P

$\neg R$

$\neg\neg Q$

$\neg P$ $\quad\quad\quad\quad$ $Q \rightarrow R$

X $\quad\quad$ $\neg Q$ $\quad\quad$ R

X $\quad\quad$ X $\quad\quad$ Valid

5. $P \wedge Q$

$\neg(Q \vee R)$

P

Q

$\neg Q$

R

X $\quad\quad$ Valid

6. $\neg(P \rightarrow Q)$

$\neg(\neg Q \vee R)$

$\neg\neg Q$

$\neg R$

P

$\neg Q$

X $\quad\quad$ Valid

7. $P \vee \neg Q$

$\neg P \vee R$

$\neg(\neg R \rightarrow \neg P)$

$\neg R$

$\neg\neg P$

$\neg P$ $\quad\quad$ R

X $\quad\quad$ X $\quad\quad$ Valid

8. $P \rightarrow Q$

$Q \rightarrow P$

$\neg(P \equiv Q)$

$\neg P$ $\quad\quad\quad\quad$ P

Q $\quad\quad\quad\quad$ $\neg Q$

$\neg P$ $\quad\quad$ Q $\quad\quad$ $\neg P$ $\quad\quad$ Q

$\neg Q \quad P \quad \neg Q \quad P \qquad X \qquad X$

$X \quad X \quad X \quad X$ Valid

9.

$P \wedge Q$

$Q \wedge \neg R$

$\neg(R \rightarrow \neg P)$

P

Q

$\neg Q$

$\neg R$

R

$\neg\neg P$

X Valid

10.

$\neg[(P \vee \neg P) \rightarrow (Q \rightarrow R)]$

$P \vee \neg P$

$\neg(Q \rightarrow R)$

Q

$\neg R$

$P \qquad \neg P$ Invalid

Explain how the truth-tree method tests for validity.

A: A truth tree is essentially a consistency test. By stacking the affirmation of the premises on top of the negation of the conclusion, the truth tree method will allow us to determine whether true premises and a false conclusion are consistent with each other. If so, then the argument is invalid. If not, then the argument is valid.

Chapter 10: Rules of Inference

1. 1) $P \rightarrow (Q \rightarrow R)$
   2) $\neg(Q \rightarrow R)$
   3) $(\neg P \vee S) \rightarrow T$ / T
   4) $\neg P$ 1,2 MP
   5) $\neg P \vee S$ 4 Add
   6) T 3,5 MP

2. 1) $P \rightarrow \neg Q$
   2) $\neg Q \rightarrow R$

3) T
4) $[(P \to R) \land T] \to S$ / S
5) $P \to R$ 1,2 HS
6) $(P \to R) \land T$ 5,3 Conj
7) S 6,4 MP

3. 1) $P \lor Q$
2) $\neg P$
3) $Q \to S$
4) $\neg R \to \neg T$
5) T / $R \land S$
6) Q 1,2 DS
7) S 3,6 MP
8) R 4,5 MT

4. 1) P
2) Q
3) $\neg(P \land Q) \lor R$
4) $R \to (S \to T)$
5) $(S \to T) \to W$ / W
6) $P \land Q$ 1,2 Conj
7) R 3,6 DS
8) $R \to W$ 4,5 HS
9) W 7,8 MP

5. 1) $P \land Q$
2) $P \to R$
3) $Q \to S$
4) $(R \land S) \to (W \lor T)$ / $W \lor T$
5) P 1 Simp
6) Q 1 Simp
7) R 2,5 MP
8) S 3,6 MP
9) $R \land S$ 7,8 Conj
10) $W \lor T$ 4,9 MP

6.	1) $P \rightarrow Q$	
	2) $\neg Q \wedge R$	
	3) $\neg P \rightarrow S$	
	4) $R \rightarrow V$	/ $S \wedge V$
	5) $\neg Q$	2 Simp
	6) R	2 Simp
	7) $\neg P$	1,5 MT
	8) S	3,7 MP
	9) V	4,6 MP
	10) $S \wedge V$	8,9 Conj

7.	1) $P \vee R$	
	2) $\neg P \wedge S$	
	3) $(R \wedge S) \rightarrow \neg V$	
	4) $V \vee Z$	/ Z
	5) $\neg P$	2 Simp
	6) R	1,5 DS
	7) S	2 Simp
	8) $R \wedge S$	6,7 Conj
	9) $\neg V$	3,8 MP
	10) Z	4,9 DS

8.	1) $P \wedge \neg S$	
	2) $P \rightarrow (S \vee V)$	
	3) $V \rightarrow Z$	/ Z
	4) P	1 Simp
	5) $S \vee V$	2,4 MP
	6) $\neg S$	1 Simp
	7) V	5,6 DS
	8) Z	3,7 MP

9.	1) P	

2) $(P \vee Q) \to \neg Z$
3) $Z \vee V$
4) $V \to R$ / R
5) $P \vee Q$ 1 Add
6) $\neg Z$ 2,5 MP
7) V 3,6 DS
8) R 4,7 MP

10. 1) $P \to Z$
2) $\neg Z \wedge V$
3) $(\neg P \wedge V) \to R$ / R
4) $\neg Z$ 2 Simp
5) $\neg P$ 1,4 MT
6) V 2 Simp
7) $\neg P \wedge V$ 5,6 Conj
8) R 3,7 MP

Chapter 11: Rules of Replacement

1. 1) $\neg R \vee S$
2) $(P \vee Q) \to (R \wedge \neg S)$ / $\neg P$
3) $\neg(R \wedge \neg S)$ 1 DM
4) $\neg(P \vee Q)$ 2,3 MT
5) $\neg P \wedge \neg Q$ 4 DM
6) $\neg P$ 5 Simp

2. 1) $P \wedge (Q \to R)$

2) $\neg(R \wedge P)$ / $\neg Q$
3) P 1 Simp
4) $\neg R \vee \neg P$ 2 DM
5) $\neg P \vee \neg R$ 4 Com
6) $\neg\neg P$ 3 DN
7) $\neg R$ 5,6 DS
8) $(Q \rightarrow R) \wedge P$ 1 Com
9) $Q \rightarrow R$ 8 Simp
10) $\neg Q$ 9,7 MT

3. 1) $H \rightarrow K$
2) $C \equiv D$
3) $\neg C \rightarrow \neg K$ / $H \rightarrow D$
4) $(C \rightarrow D) \wedge (D \rightarrow C)$ 2 Equiv
5) $C \rightarrow D$ 4 Simp
6) $K \rightarrow C$ 3 Cont
7) $K \rightarrow D$ 6,5 HS
8) $H \rightarrow D$ 1,7 HS

4. 1) $\neg P \rightarrow (Q \wedge R)$
2) $\neg R$ / P
3) $\neg(Q \wedge R) \rightarrow \neg\neg P$ 1 Cont
4) $(\neg Q \vee \neg R) \rightarrow \neg\neg P$ 3 DM
5) $\neg R \vee \neg Q$ 2 Add
6) $\neg Q \vee \neg R$ 5 Com
7) $\neg\neg P$ 4,6 MP
8) P 7 DN

5. 1) $\neg R \vee \neg S$
2) $P \rightarrow (R \wedge S)$ / $\neg P$
3) $\neg(R \wedge S)$ 1 DM
4) $\neg P$ 2,3 MT

6. 1) $\neg P$
2) $(P \vee Q) \equiv R$
3) $\neg Q$ / $\neg(R \wedge S)$
4) $[(P \vee Q) \wedge R] \vee [\neg(P \vee Q) \wedge \neg R]$ 2 Equiv
5) $\neg P \wedge \neg Q$ 1,3 Conj

6) $\neg(P \lor Q)$	5 DM
7) $\neg(P \lor Q) \lor \neg R$	6 Add
8) $\neg[(P \lor Q) \land R]$	7 DM
9) $\neg(P \lor Q) \land \neg R$	4,8 DS
10) $\neg R \land \neg(P \lor Q)$	9 Com
11) $\neg R$	10 Simp
12) $\neg R \lor \neg S$	11 Add
13) $\neg(R \land S)$	12 DM

7.

1) $P \to Q$	
2) $(Q \land R) \to S$	/ $(Q \to (R \to S)$
3) $Q \to (R \to S)$	2 Exp

8. 1) $P \lor (Q \land R)$

2) $\neg R$	/ P
3) $(P \lor Q) \land (P \lor R)$	1 Dist
4) $(P \lor R) \land (P \lor Q)$	3 Com
5) $P \lor R$	4 Simp
6) $R \lor P$	5 Com
7) P	6,2 DS

CH 12: Proof Rules

1.

1) $(R \lor S) \to T$	
2) $(P \lor Q) \to T$	
3) $R \lor P$	/ T
4) $\neg T$	AIP
5) $\neg(R \lor S)$	1,4 MT
6) $\neg(P \lor Q)$	2,4 MT
7) $\neg R \land \neg S$	5 DM
8) $\neg P \land \neg Q$	6 DM
9) $\neg R$	7 Simp

	10) $\neg P$	8 Simp
	11) P	3,9 DS
	12) $P \wedge \neg P$	10,11 Conj
	13) T	4-12 IP
2.	1) $[P \to (Q \to P)] \to S$	/ S
	2) $\neg S$	AIP
	3) $\neg [P \to (Q \to P)]$	1,2 MT
	4) $\neg [\neg P \vee (Q \to P)]$	3 Impl
	5) $\neg \neg P \wedge \neg (Q \to P)$	4 DM
	6) $\neg \neg P$	5 Simp
	7) $\neg (Q \to P) \wedge \neg \neg P$	5 Com
	8) $\neg (Q \to P)$	7 Simp
	9) $\neg (\neg Q \vee P)$	8 Impl
	10) $\neg \neg Q \wedge \neg P$	9 DM
	11) $\neg P \wedge \neg \neg Q$	10 Com
	12) $\neg P$	11 Simp
	13) $\neg P \wedge \neg \neg P$	12,6 Conj
	14) S	2-13 IP
3.	1) $(S \vee T) \to \neg S$	/ $\neg S$
	2) $\neg \neg S$	AIP
	3) $\neg (S \vee T)$	1,2 MT
	4) $\neg S \wedge \neg T$	3 DM
	5) $\neg S$	4 Simp
	6) $\neg S \wedge \neg \neg S$	5,2 Conj
	7) $\neg S$	2-6 IP
4.	1) $(P \to P) \to R$	
	2) $(R \vee S) \to Q$	/ Q
	3) $\neg Q$	AIP
	4) $\neg (R \vee S)$	2,3 MT
	5) $\neg R \wedge \neg S$	4 DM
	6) $\neg R$	5 Simp
	7) $\neg (P \to P)$	1,6 MT
	8) $\neg (\neg P \vee P)$	7 Impl
	9) $\neg \neg P \wedge \neg P$	8 DM

10) Q	3-9 IP

5.

1) $P \to (Q \lor R)$	
2) $(P \to R) \to (S \land T)$	
3) $Q \to R$	/ T
4) $\neg T$	AIP
5) $\neg T \lor \neg S$	4 Add
6) $\neg S \lor \neg T$	5 Com
7) $\neg(S \land T)$	6 DM
8) $\neg(P \to R)$	2,7 MT
9) $\neg(\neg P \lor R)$	8 Impl
10) $\neg\neg P \land \neg R$	9 DM
11) $\neg\neg P$	10 Simp
12) P	11 DN
13) $Q \lor R$	1,12 MP
14) $\neg R \land \neg\neg P$	10 Com
15) $\neg R$	14 Simp
16) $R \lor Q$	13 Com
17) Q	15,16 DS
18) $\neg Q$	3,15 MT
19) $Q \land \neg Q$	17,18 Conj
20) T	4-19 IP

6.

1) $Q \to (R \to S)$	
2) $Q \to (T \to \neg U)$	
3) $U \to (R \lor T)$	/ $Q \to (U \to S)$
4) Q	ACP
5) $R \to S$	1,4 MP
6) $T \to \neg U$	2,4 MP
7) U	ACP
8) $U \to \neg T$	6 Cont
9) $\neg T$	7,8 MP
10) $R \lor T$	3,7 MP
11) $T \lor R$	10 Com

12) R	9,11 DS
13) S	5,12 MP
14) $U \to S$	7-13 CP
15) $Q \to (U \to S)$	4-14 CP

7.

1) $P \to Q$	
2) $P \to R$	/ $P \to (Q \wedge R)$
3) P	ACP
4) Q	1,3 MP
5) R	2,3 MP
6) $Q \wedge R$	4,5 Conj
7) $P \to (Q \wedge R)$	3-6 CP

8.

1) $\neg(U \wedge W) \to X$	
2) $U \to \neg U$	/ $\neg(U \vee \neg X)$
3) $U \vee \neg X$	AIP
4) $\neg U \vee \neg U$	2 Impl
5) $\neg U$	4 Taut
6) $\neg X$	3,5 DS
7) $\neg U \vee \neg W$	5 Add
8) $\neg(U \wedge W)$	7 DM
9) X	1,8 MP
10) $X \wedge \neg X$	9,6 Conj
11) $\neg(U \vee \neg X)$	3-10 IP